1 Vector Spaces

Since Linear Algebra is the study of linear transformations on vector spaces, the topic of vector spaces is certainly a reasonable place to start. One could easily argue that it's actually the only reasonable place to start.[1] As we saw in the previous chapter, you cannot construct a function without clearly defining its domain. Linear transformations are functions, and we'll eventually see that vector spaces are used as their domains. Thus, we should begin here with one of the two main characters of this story, vector spaces.

1: Wha? Oh, yes. That *is* my gauntlet on the ground!

1.1 Vector Spaces

$\mathbb{R}$? Yep, $\mathbb{R}$. It turns out the set of real numbers, $\mathbb{R}$, is a vector space. This seems to be an excellent place to start our discussion.

There are many ways to construct the real numbers.[2] That said, we're not going to spend any time constructing the real numbers, $\mathbb{R}$, any more than you would in a typical calculus course. We're going to step over this particular mathematical duck by simply saying the following dissatisfying thing:

2: They are all super interesting, very cool, and usually discussed in a Real Analysis course.

$$\mathbb{R} = (-\infty, \infty). \tag{1.1}$$

Note that the authors have chosen to number this equation. This is so that if future equations seem mystifying, you, the reader, can be referred back to this thing for some perspective.

The goal of this section is not to define $\mathbb{R}$ but to understand the properties of $\mathbb{R}$ with typical operations (adding and multiplying) that we really like. Hey, speaking of operations, let's recall some in case any of you skipped over Chapter 0.

Definition 1.1.1 ***Addition*** *is the function* $(+)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by relating two real numbers to their sum.* ***Multiplication*** *is the function* $(\cdot)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by relating two real numbers to their product.*

You should be very familiar with both of these operations and all of the properties they enjoy; you probably know them all by name. Nevertheless, here's a nice comprehensive list:

Theorem 1.1.1 (Field Axioms for Real Numbers) *Let $+$ and $\cdot$ be used for the usual operations of addition and multiplication, respectively. There are elements $0, 1 \in \mathbb{R}$ such that for any $a, b, c \in \mathbb{R}$ and any nonzero $d \in \mathbb{R}$,*

Commutativity of Addition	$a + b = b + a$,
Associativity of Addition	$(a + b) + c = a + (b + c)$,
Additive Identity	$a + 0 = 0 + a = a$,
Additive Inverses	*there is an element $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$,*
Commutativity of Multiplication	$a \cdot b = b \cdot a$,
Associativity of Multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
Multiplicative Identity	$a \cdot 1 = 1 \cdot a = a$,
Multiplicative Inverses	*there is an element $d^{-1} \in \mathbb{R}$ such that $d \cdot d^{-1} = d^{-1} \cdot d = 1$, and*
Distributive Property	$a \cdot (b + c) = a \cdot b + a \cdot c$.

The real numbers are pretty great, though. We built a whole sequence of calculus courses using them! Asking that any old set do all nine of these things is a bit much. Let's see what happens when we change it up a bit.

Example 1.1.1 Let's first consider the set $\mathbb{Z}$. Which of these properties hold and which fail for $\mathbb{Z}$? First of all, since we're using a different set ($\mathbb{Z}$ instead of $\mathbb{R}$), we need to be sure that $\mathbb{Z}$ is closed under the operations of addition and multiplication as well; indeed, $\mathbb{Z}$ is closed under addition and multiplication. Moreover, since these are actually the same operations as the ones in $\mathbb{R}$, the following properties hold in $\mathbb{Z}$ because they hold in $\mathbb{R}$:

- Commutativity of Addition: $a + b = b + a$ for $a, b \in \mathbb{Z}$
- Associativity of Addition: $(a+b)+c = a+(b+c)$ for $a, b, c \in \mathbb{Z}$
- Commutativity of Multiplication: $a \cdot b = b \cdot a$ for $a, b \in \mathbb{Z}$
- Associativity of Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for $a, b, c \in \mathbb{Z}$
- Distributive Property: $a \cdot (b + c) = a \cdot b + a \cdot c$ for $a, b, c \in \mathbb{Z}$

These were all properties of the operations $+$ and $\cdot$ instead of properties of the set. The properties of the set will require a bit more thought from us.

- Additive Identity: Since this is the same addition as for $\mathbb{R}$, our element 0 is still the additive identity. The condition we are now checking is whether this element is in $\mathbb{Z}$, which it is! Thus, this one also holds for $\mathbb{Z}$.
- Additive Inverses: Again, since any $a \in \mathbb{Z}$ is also in $\mathbb{R}$, the additive inverse of a is still the same as in $\mathbb{R}$, so for any $a \in \mathbb{Z}$, we need to see that $-a \in \mathbb{Z}$, which it is! Thus, this one also holds for $\mathbb{Z}$.
- Multiplicative Identity: Same as these others, we need only note that $1 \in \mathbb{Z}$ to see this one holds as well.

- Multiplicative Inverses: Again, the multiplicative inverses are the same as the ones in $\mathbb{R}$, so we need only check that if $a \in \mathbb{Z}$, then $\frac{1}{a} \in \mathbb{Z}$. But this one fails! In particular, $2 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

Exploration 10 Can you name another set with the operations of addition and multiplication where all these properties hold? (Hint: Many answers do exist.)

Exploration 11 Find a property that fails for $\mathbb{R}$ if we replace $+$ with $-$.

The real numbers work super well. Having two operations is pretty great. We get all those extra rules to make sure they both work correctly and play nice together. We get to put them in a list and memorize them. It's all very satisfying.

Let's scrap it all and start over. Don't memorize anything yet. Well, maybe we'll keep that addition bit. Most people rather like that one. But multiplication? Be honest. Hasn't "repeated addition" always felt a little scammy to you as an operation?[3]

3:  Total pyramid scheme!

Vector Spaces, by Definition

In all seriousness, though, what was just described is roughly what's about to happen. The set of real numbers is a very complicated set. You do get a lot of very nice properties with addition and multiplication on $\mathbb{R}$, but the properties themselves, when taken all together, can be restrictive in some ways. The idea is to give up some of the properties we enjoyed from Theorem 1.1.1 so that we can enjoy that smaller collection of properties on a wider variety of sets.

It's not as bad as it sounds. In fact, we'll still have two operations!

Definition 1.1.2 *A* ***vector space*** *over $\mathbb{R}$ is a set V (whose elements we call* ***vectors****) together with two operations that satisfies all the properties listed below.*

- *The first operation, called* ***vector addition*** $(+)\colon V \times V \to V$, *relates two vectors $\vec{v}$ and $\vec{w}$ to a third vector, commonly written as $\vec{v}+\vec{w}$, and called the sum of these two vectors.*
- *The second operation, called* ***scalar multiplication*** $(\cdot)\colon \mathbb{R} \times V \to V$ *relates any scalar $a \in \mathbb{R}$ and any vector $\vec{v} \in V$ to another vector $a\vec{v}$.*

There are elements $\vec{0} \in V$ and $1 \in \mathbb{R}$ such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all $a, b \in \mathbb{R}$,

Closure under Addition	$\vec{u}+\vec{v} \in V$,
Associativity of Addition	$\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$,
Commutativity of Addition	$\vec{u}+\vec{v} = \vec{v}+\vec{u}$,
Additive Identity	$\vec{v}+\vec{0} = \vec{0}+\vec{v} = \vec{v}$,
Additive Inverses	*there exists an element* $-\vec{v} \in V$ *such that* $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$,
Closure under Scalar Multiplication	$a\vec{v} \in V$,
Scalar and Real Multiplication	$a(b\vec{v}) = (ab)\vec{v}$,
Multiplicative Identity	$1\vec{v} = \vec{v}$,
Distributivity Over Vector Addition	$a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$, *and*
Distributivity Over Real Addition	$(a+b)\vec{v} = a\vec{v}+b\vec{v}$.

Note that before you can even call something a vector space and start talking about all the cool stuff it can or can't do, you need a set and two operations. Therefore, if someone says to you, "Hey! Check out this cool set! I think it's a vector space." Your immediate reaction should be roughly, "Oh yeah? With what operations?" If the other person runs away, they were probably wrong about their set. If they produce two reasonable operations that could be called vector addition and scalar multiplication, then you've found a worthy companion to assist in verifying every single one of the axioms neatly listed in Definition 1.1.2. Then, and only then, should you dare declare your set a vector space.

Fun fact: $\mathbb{R}$ is a vector space.[4] This is not surprising at all; it was, after all, the muse which begat this fun new definition.[5] Now, let's take a bunch of Cartesian products!

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$$

People use all sorts of goofy notaton for $\mathbb{R}^n$. The preceding one is respectable enough. Your calculus book might use something like

$$\mathbb{R}^n = \{\langle x_1, \ldots, x_n\rangle : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}.$$

We assume the common use of this notation is due to its pointy-ness and general sinister appearance. We'll more often use the seemingly obnoxious notation

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n \right\}. \tag{1.2}$$

What a neat set! Did you know that $\mathbb{R}^n$ is a vector space?

Ahem.

We're waiting…

Fun fact: If there's some other field you prefer, you can use that one instead of $\mathbb{R}$ here, and define vector spaces over that field instead. Wild, right? Anywho, we're gonna stick with vector spaces over $\mathbb{R}$ in this text.

Vector spaces over other fields are often used in physics and computer science and are a topic in abstract algebra courses-

-ahem! We're gonna stick with vector spaces over $\mathbb{R}$ in this text.

4: Oh yeah? With what operations?

Addition and multiplication. You know, the regular ones.

5: Yes,"begat." It's a perfectly good word. Challenge: Find a way to use this word in a sentence of your own today.

Oh? Operations? Yes, of course. Vector addition is done componentwise. For any $\vec{x}, \vec{y} \in \mathbb{R}^n$, where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \text{ we define } \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}. \tag{1.3}$$

For scalar multiplication, just multiply the scalar to each component of the given vector. That is, for any $\vec{x} \in \mathbb{R}^n$ and any $a \in \mathbb{R}$, we define

$$a\vec{x} = a \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix}. \tag{1.4}$$

Very nice, eh? Perfectly good vector addition and scalar multiplication. We defined these using what's called a *general element* of $\mathbb{R}^n$. We should talk more about this.

Many people find the most difficult part in verifying the axioms of a vector space (besides the tedium) to be showing each axiom works *for all* potential vectors in the given potential vector space. Almost all the real vector spaces we will see end up having an infinite number of vectors, so verifying each property for literally all vectors is not feasible. What one needs is a *general vector* that could represent any vector in the set. Nothing too specific, this vector needs to be generic enough that it satisfies all the properties required to be in the set *and nothing else*. Let's look at $\mathbb{R}^3$.

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, 3 \right\} \tag{1.5}$$

Now what does it mean exactly to be "in $\mathbb{R}^3$?" While we've settled on the notation above, we also saw two others. Specifically,

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) \colon x_1, x_2, x_3 \in \mathbb{R}\} \text{ or } \mathbb{R}^3 = \{\langle x_1, x_2, x_3 \rangle \colon x_1, x_2, x_3 \in \mathbb{R}\}.$$

It's instructive now to consider what they all have in common. In each notation, we would use three real numbers; each could be any real number, so we used a variable x_i (where $1 \le i \le 3$) for each. These sequential subscripts also suggest these entries are ordered. That's a lot of information that's easy to look past! Evidently, whether we write them vertically or horizontally does not matter; we choose the former for reasons that will be clear later.

To make a general vector in $\mathbb{R}^3$, we need three real *variables* in order. Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is a general vector in } \mathbb{R}^3;$$

$$\begin{bmatrix} 8 \\ -1.3 \\ 1/2 \end{bmatrix} \text{ is not. It is a \textit{specific} vector in } \mathbb{R}^3.$$

Most of the axioms in Definition 1.1.2 involve more than one vector, so we'd actually need two general vectors. It's standard to just use a different letter for the variable, so for $\vec{x}, \vec{y} \in \mathbb{R}^3$, we would write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

These are both general vectors in $\mathbb{R}^3$. If we can show something is true with these, then it can be assumed to be true for *any* vector in $\mathbb{R}^3$. It's sometimes not hard at all to find a general vector for a vector space; they are often given in the definition of the set, as in Equation 1.5.

Let's show that $\mathbb{R}^2$ satisfies all the requirements to be a vector space given in Definition 1.1.2.[6] We'll make careful use of general vectors to do so.

6: We could just as easily show $\mathbb{R}^3$ or $\mathbb{R}^{100}$, but it'll take up less space if we do $\mathbb{R}^2$.

Example 1.1.2 Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. Then,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \text{ and } \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

for some $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$.

- **Closure under Addition:** Note that

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}.$$

  Since $x_i, y_i \in \mathbb{R}$ for each $i = 1, 2$ and we know $\mathbb{R}$ is closed under addition, each $x_i + y_i \in \mathbb{R}$. Thus, $\vec{x} + \vec{y} \in \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under this vector addition.

- **Associativity of Addition:**

$$\begin{aligned} \vec{x} + (\vec{y} + \vec{z}) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \end{bmatrix} = \begin{bmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \end{bmatrix} \\ &= \begin{bmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2) + z_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= (\vec{x} + \vec{y}) + \vec{z}. \end{aligned}$$

  Here we used the associative property for $\mathbb{R}$ in each component.

- **Commutativity of Addition:**

$$\begin{aligned} \vec{x} + \vec{y} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{y} + \vec{x} \end{aligned}$$

  Here, we used the commutative property of real numbers to see that $x_1 + y_1 = y_1 + x_1$ and $x_2 + y_2 = y_2 + x_2$.

- **Additive Identity:** For this, we need to identify the additive identity $\vec{0}$, the special vector in $\mathbb{R}^2$ such that $\vec{x} + \vec{0} = \vec{x}$ for any $\vec{x} \in \mathbb{R}^2$. There's something pretty obvious to guess, but that won't always be the case. We'll do this one the "long way." Suppose our identity is the vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then we know $\vec{x} + \vec{v} = \vec{x}$, which means

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We can use the definition of our vector addition then to get

$$\begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

so $x_1 + v_1 = x_1$, which says $v_1 = 0$. Similarly, $x_2 + v_2 = x_2$, so $v_2 = 0$ as well. This says our $\vec{v} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Fortunately, $\vec{0} \in \mathbb{R}^2$ and this calculation works for any vector $\vec{x}$, so our set contains an additive identity!

- **Additive Inverses:** Now that we know what the additive identity is, we can find the additive inverse of a vector $\vec{x} \in \mathbb{R}^2$. This is the special vector $\vec{v}$ such that $\vec{x} + \vec{v} = \vec{0}$, so if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then we see that

$$\begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $x_1 + v_1 = 0$ and $v_1 = -x_1$. We also see $v_2 = -x_2$, so the additive inverse of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is $\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$, which is also in $\mathbb{R}^2$. Thus, every element in $\mathbb{R}^2$ has an additive inverse in $\mathbb{R}^2$.

- **Closure under Scalar Multiplication:** We need to argue why $a\vec{x} \in \mathbb{R}^2$ for any real number a and any vector $\vec{x} \in \mathbb{R}^2$. Note that

$$a\vec{x} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}.$$

Then since $a, x_1, x_2 \in \mathbb{R}$ and $\mathbb{R}$ is closed under multiplication, we know $ax_1, ax_2 \in \mathbb{R}$. This means $a\vec{x} \in \mathbb{R}^2$.

- **Scalar and Real Multiplication:** This property is essentially guaranteeing our definition of scalar multiplication works well with the usual idea of multiplication in $\mathbb{R}$. Formally, we need to show $(ab)\vec{x} = a(b\vec{x})$. Using the definition of scalar multiplication and the associative property for multiplication of real numbers, we have

$$(ab)\vec{x} = (ab)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \end{bmatrix} = \begin{bmatrix} a(bx_1) \\ a(bx_2) \end{bmatrix}$$
$$= a\begin{bmatrix} bx_1 \\ bx_2 \end{bmatrix} = a\left(b\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = a(b\vec{x}).$$

- **Multiplicative Identity:** In the case of the additive identity, we had to find it and be sure it was in the set. Here, we know the multiplicative identity has to be 1, but we just need to check that it does what it should with this scalar multiplication definition. Let's see what $1\vec{x}$ is then.

$$1\vec{x} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 \\ 1x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}.$$

Yay! It works!

- **Distributivity Over Vector Addition:** For this, we are checking that scalar multiplication distributes across vector addition, so we need to see $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$ for any real number a and any vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$. Using the distributive property for real numbers, we get

$$\begin{aligned} a(\vec{x}+\vec{y}) &= a\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = a\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} \\ &= \begin{bmatrix} a(x_1+y_1) \\ a(x_2+y_2) \end{bmatrix} = \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = a\vec{x}+a\vec{y}. \end{aligned}$$

- **Distributivity Over Real Addition:** This last axiom is checking that our new operations work well with our classic addition in $\mathbb{R}$. That is, we need to verify that $(a+b)\vec{x} = a\vec{x}+b\vec{x}$ for any real numbers a and b and any vector $\vec{x} \in \mathbb{R}^2$. Again using the distributive property for real numbers, we get

$$\begin{aligned} (a+b)\vec{x} &= (a+b)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \end{bmatrix} = \begin{bmatrix} ax_1+bx_1 \\ ax_2+bx_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \end{bmatrix} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\vec{x}+b\vec{x}. \end{aligned}$$

Great! Now, you've seen how a set with operations of vector addition and scalar multiplication can be shown to be a vector space! Note that for each of the axioms, we relied upon established knowledge about $\mathbb{R}$, the set of real numbers.[7] However, we only showed $\mathbb{R}^2$ is a vector space. We *claimed* that $\mathbb{R}^n$ is a vector space for any positive integer n. Well, that proof is very similar. Let's see how by focusing on closure of addition for $\mathbb{R}^n$. That is, given any $\vec{x}, \vec{y} \in \mathbb{R}^n$, we want to show $\vec{x}+\vec{y} \in \mathbb{R}^n$. Note that

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are general vectors in $\mathbb{R}^n$. Then by definition of vector addition in $\mathbb{R}^n$ (Equation 1.3), we have

$$\vec{x}+\vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}. \tag{1.6}$$

It remains to show that the vector at the end of Equation 1.6 is a vector in $\mathbb{R}^n$. Since $\mathbb{R}$ is closed under addition, we know that $x_i + y_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Thus,

$$\begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} \in \mathbb{R}^n.$$

It follows that $\vec{x}+\vec{y} \in \mathbb{R}^n$, so $\mathbb{R}^n$ is closed under vector addition.

7: This is what we do in mathematics! We build logically upon what is already known.

Exploration 12 Following arguments similar to Example 1.1.2 and the one above, show that $\mathbb{R}^n$ is a vector space when n is any positive integer. Well, that might take a while. At least show two of the axioms hold, so you get a feel for how this goes.

Remark: There is a small hiccup with your favorite[8] vector space, $\mathbb{R}$. Since we're usually dealing with vector spaces over $\mathbb{R}$, when we talk about the vector space $\mathbb{R}$, it can be unclear whether a real number is a vector or a scalar. We resolve this ambiguity by using bold font for vectors, so if you see $\vec{5} \in \mathbb{R}$, you should interpret this as a vector in the vector space $\mathbb{R}$. If you see $5 \in \mathbb{R}$, you should interpret this as the scalar 5. Thus, if you want to rescale the vector $\vec{5} \in \mathbb{R}$ by the scalar 5, you would have

$$5\vec{5} = \vec{25} \in \mathbb{R}.$$

Exactly what just happened there? Well, the vector $\vec{5}$ rescaled by the scalar 5 gives us the vector $\vec{25}$. It's potentially confusing because the real number 25 is being interpreted as a vector. We concede this is both weird and annoying, but it cannot be avoided. Again, *we use bold font to indicate when mathematical objects are vectors*; this will always serve to answer whether or not something is a vector.[9]

8: My favorite is actually $\mathbb{R}^2$.

9: Unless there is a typo. No guarantees.

Other Vector Spaces

One of the most powerful aspects of vector spaces is the wide variety of sets that can be made into vector spaces by providing appropriate operations. We'll now discuss an example of a standard vector space that is not $\mathbb{R}^n$, and you'll see other examples in the exercises.

The set of polynomials of degree n or less is also a very nice set we can make into a vector space:

$$\mathbb{P}_n = \{a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}. \tag{1.7}$$

The "vectors" in this set are polynomials, so we often write strange-looking things like

$$\vec{p} = 27 - 2x + x^n.$$

Here we're saying that $\vec{p}$ is the polynomial in the set $\{a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}$, where $a_0 = 27$, $a_1 = -2$, $a_3 = \cdots = a_{n-1} = 0$, and $a_n = 1$.

Given general $\vec{p}, \vec{q} \in \mathbb{P}_n$, which we may write as

$$\begin{aligned}\vec{p} &= a_0 + a_1x + \cdots a_nx^n \text{ and}\\ \vec{q} &= b_0 + b_1x + \cdots b_nx^n,\end{aligned}$$

where $a_i, b_i \in \mathbb{R}$, and a scalar $c \in \mathbb{R}$, we define vector addition as the usual polynomial addition

$$\begin{aligned}\vec{p} + \vec{q} &= (a_0 + a_1x + \cdots a_nx^n) + (b_0 + b_1x + \cdots b_nx^n)\\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n,\end{aligned}$$

and we define scalar multiplication as

$$c\vec{p} = c(a_0 + a_1x + \cdots a_nx^n) = ca_0 + ca_1x + \cdots + ca_nx^n.$$

Example 1.1.3 We won't spoil all the fun, but let's show $\mathbb{P}_n$ satisfies at least one of the axioms to be a vector space with these operations.

- **Commutativity of Vector Addition:** Let $\vec{p}, \vec{q} \in \mathbb{P}_n$ for some positive integer n. Then

  $\vec{p} = a_0 + a_1x + \cdots a_nx^n$ and $\vec{q} = b_0 + b_1x + \cdots b_nx^n$

  for some real numbers $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$. Using the definitions above and the property of commutativity for addition of real numbers, we have

$$\begin{aligned}\vec{p} + \vec{q} &= [a_0 + a_1x + \cdots a_nx^n] + [b_0 + b_1x + \cdots b_nx^n]\\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n\\ &= (b_0 + a_0) + (b_1 + a_1)x + \cdots + (b_n + a_n)x^n\\ &= [b_0 + b_1x + \cdots b_nx^n] + [a_0 + a_1x + \cdots a_nx^n] = \vec{q} + \vec{p}.\end{aligned}$$

  Yay! Vector addition commutes!

Exploration 13 We should really verify all the axioms in Definition 1.1.2 for $\mathbb{P}_n$. We've done one together, so that leaves nine more! For the sake of brevity, though, let's walk through a modified version of this exercise.

- What is the additive identity in $\mathbb{P}_n$?

- What is the additive inverse of the polynomial $5x - 3x^3 + 4x^7$ in $\mathbb{P}_8$?

- What is the additive inverse of $\vec{p} = a_0 + a_1x + \cdots a_nx^n$ in $\mathbb{P}^n$?

- Show that $12(5x-3x^3+4x^7) = 2(6(5x-3x^3+4x^7))$ by simplifying both sides separately.

- Following the specific example above, show that $ab\vec{p} = a(b\vec{p})$ for any $\vec{p} \in \mathbb{P}_n$ and any $a, b \in \mathbb{R}$.

Great! That's a total of four of the axioms down. Just six more to go!

Note that in the exploration above, we sometimes asked that you work a specific example before taking on the task of verifying the more general statement. This is a worthwhile problem-solving strategy, and you may find it a useful starting place whenever you are tasked thusly. Just be sure that you do not confuse it with showing the general statement. *An affirmative example will not let you conclude the statement holds in general*. And yet, what about when the statement does not hold?

Example 1.1.4 We have now seen some examples of sets with operations of vector addition and scalar multiplication that we've at least claimed are vector spaces. What does it look like when the set is *not* a vector space? Consider again the set $\mathbb{Z}$ of integers. We can use the regular addition of integers for our vector addition and use multiplication of real numbers for our scalar multiplication. This is not a vector space. Let's see a property that fails:

- **Closure under Scalar Multiplication:** When we multiply an integer by a real number, we have no reason to expect the outcome need be an integer. While this statement is true, a much stronger statement would involve an actual example where this property fails. Consider $2 \in \mathbb{Z}$ and $1/3 \in \mathbb{R}$. Note that $(1/3)\mathbf{2} \notin \mathbb{Z}$, so $\mathbb{Z}$ is not closed under this scalar multiplication.

The example of failure used above is a technique called "providing a counterexample," and it is a very useful way to show one of our properties of a vector space fails.

Section Highlights

This section is where we encounter one of the two main topics of this text for the first time, a *vector space*.

- A *vector space* is a set with elements we call *vectors* together with the operations of *vector addition* and *scalar multiplication*.
- There are 10 properties that a set and operations must satisfy in order for it to be a vector space. See Definition 1.1.2.
- To show one of these properties holds, one must use a general elements from the set. See Example 1.1.2.
- To show one of these properties fails, a specific example of it failing using elements from the set is sufficient. See Example 1.1.4.

- A few important examples of real vector spaces:
  $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{P}_n$ for any positive integer, n.

Exercises for Section 1.1

1.1.1. For each of the 10 vector space properties, define an operation of vector addition and/or scalar multiplication on $\mathbb{R}$ for which the property fails. Give an explicit example to show the failure.

1.1.2. Give the complete argument that $\mathbb{P}_n$ is a vector space.

1.1.3. The complex numbers $\mathbb{C}$ are $\{a+bi\colon a,b\in\mathbb{R}\}$ where $i=\sqrt{-1}$. Verify all the vector space axioms to show that $\mathbb{C}$ is a vector space over the field $\mathbb{R}$.

1.1.4. Below are several operations on $\mathbb{R}$. To keep from confusing them with the standard operations, we'll use the symbol $\boxplus$ to denote them. Determine whether these operations obey the commutative and associative properties.

(a) $a \boxplus b = a + 2b$ for any $a, b \in \mathbb{R}$

(b) $a \boxplus b = ab$ for any $a, b \in \mathbb{R}$

(c) $a \boxplus b = a + b + ab$ for any $a, b \in \mathbb{R}$

(d) $a \boxplus b = a + b - 3$ for any $a, b \in \mathbb{R}$

(e) $a \boxplus b = a + b - ab$ for any $a, b \in \mathbb{R}$

1.1.5. Below are several operations on the given set V. To keep from confusing them with the standard operations, we'll use the symbol $\boxplus$ to denote them. For each of these operations, determine whether there is an additive identity (a.k.a zero vector) $\vec{z}$ such that $\vec{x} \boxplus \vec{z} = \vec{z} \boxplus \vec{x} = \vec{x}$ for any $\vec{x} \in V$. If it does exist, what is it?

(a) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + 2b$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(b) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(c) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b + ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(d) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b - 3$ for any $\vec{a}, \vec{b} \in \mathbb{R}$.

(e) Let $V = \mathbb{R}^2$. Define

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$.

(f) Let $V = \mathbb{R}^2$. Define

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c-3 \\ b+d+4 \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$.

(g) Let $V = \mathbb{R}^2$. Define

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ b+d \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$.

1.1.6. Note that the additive inverse of $\vec{x} \in V$ with the operation $\boxplus$ is the unique element $\vec{i}$ such that $\vec{x} \boxplus \vec{i} = \vec{i} \boxplus \vec{x} = \vec{z}$ where $\vec{z}$ is the additive identity. Thus, in order for an element to have an additive inverse, the set must first have an additive identity. Use your work from the previous exercise to determine whether the additive identity for the given vectors exists.

(a) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. If possible, find the additive inverse for $\vec{5}$ and $\vec{0}$.

(b) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b + ab$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. If possible, find the additive inverse for $\vec{5}$ and $\vec{-1}$.

(c) Let $V = \mathbb{R}$. Define $\vec{a} \boxplus \vec{b} = a + b - 3$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. If possible, find the additive inverse for $\vec{5}$ and $\vec{0}$.

1.1.7. Define $\vec{a} \boxplus \vec{b} = a + b - 3$ for any $\vec{a}, \vec{b} \in \mathbb{R}$. In each of the parts below, we also define a new operation $\odot$ for our scalar multiplication. With this notation, the two distributive properties become:

Distributivity Across Vector Addition	$k \odot (\vec{x} \boxplus \vec{y}) = (k \odot \vec{x}) \boxplus (k \odot \vec{y})$
Distributivity Across Scalar Addition	$(j + k) \odot \vec{x} = (j \odot \vec{x}) \boxplus (k \odot \vec{x})$.

Verify whether these properties hold for this vector addition $\boxplus$ and the given $\odot$.

(a) Define $k \odot \vec{a} = ka$, for any $k \in \mathbb{R}$, $\vec{a} \in \mathbb{R}$.

(b) Define $k \odot \vec{a} = ka - 3$, for any $k \in \mathbb{R}$, $\vec{a} \in \mathbb{R}$.

(c) Define $k \odot \vec{a} = ka - 3k + 3$, for any $k \in \mathbb{R}$, $\vec{a} \in \mathbb{R}$.

1.1.8. Let's consider $\mathbb{R}$, with a new operation. Let's replace the usual vector addition with the operation "$\boxplus$" defined by $a \boxplus b = a + b + ab$ for any $a, b \in \mathbb{R}$. Determine whether $\mathbb{R}$ is still a vector space with the operation $\boxplus$ for vector addition and the usual scalar multiplication. If it is not, which properties fail?

1.1.9. Let's consider $\mathbb{R}$, with two new operations. Let's replace the usual vector addition with the operation "$\boxplus$" defined by $a \boxplus b = a + b - 3$ for any $a, b \in \mathbb{R}$ and scalar multiplication defined by $k \odot a = ka - 3k + 3$ for any $k \in \mathbb{R}$ and $a \in \mathbb{R}$. Determine whether $\mathbb{R}$ is still a vector space with the operation $\boxplus$ for vector addition and the operation $\odot$ for scalar multiplication. If it is not, which properties fail?

1.1.10. Let's consider $\mathbb{R}^2$ with a new operation as well. Let's vector addition with "$\boxplus$" defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix}$$

for any $a, b, c, d \in \mathbb{R}$. Determine whether $\mathbb{R}^2$ is still a vector space with the operation $\boxplus$ and the usual scalar multiplication. If it is not, which properties fail?

1.1.11. Let's consider $\mathbb{P}_2$ with a new operation. Replace vector addition with "$\boxplus$" defined by

$$(a + bx + cx^2) \boxplus (d + ex + fx^2) = (b + 2cx) + (e + 2fx) = (b + e) + 2(c + f)x.$$

Determine whether $\mathbb{P}_2$ is still a vector space with the operation $\boxplus$ and the usual scalar multiplication. If it is not, which properties fail?

1.1.12. Let $V = \{a \colon a \in \mathbb{R}, a > 0\} = (0, \infty)$. Verify V is a vector space over $\mathbb{R}$ with vector addition given by $\vec{a} \boxplus \vec{b} = \vec{ab}$ (vector addition is defined as real multiplication as positive numbers) and scalar multiplication given by $k\vec{a} = \mathbf{a^k}$ (scalar multiplication is defined as real exponentiation) for any $k \in \mathbb{R}$, $\vec{a}, \vec{b} \in V$.

1.1.13. Let $V = \{a \colon a \in \mathbb{R}, a > 0\} = (0, \infty)$. As in the previous exercise, let vector addition be given by $\vec{a} \boxplus \vec{b} = \vec{ab}$, but now, let scalar multiplication be given by $k\vec{a} = \mathbf{k^a}$ for any $k \in \mathbb{R}$, $\vec{a}, \vec{b} \in V$. Determine whether this is a vector space.

1.1.14. Let V be the interval $(0, 1)$ on the real numbers. Define vector addition as $\vec{a} \boxplus \vec{b} = \vec{ab}$ for $\vec{a}, \vec{b} \in V$ and scalar multiplication as $k\vec{a} = \mathbf{a^k}$ for any $k \in \mathbb{R}$ and $\vec{a} \in V$. This is *not* a vector space over $\mathbb{R}$. List each of the properties of a vector space that fail.

1.1.15. Let $X = \mathbb{R} \cup ☺$, where ☺ is a new number referred to as "unity face." For all $x \in X$, define $x + ☺ = ☺ + x = ☺$ and $x(☺) = ☺(x) = ☺$. With the usual addition and multiplication operations, is X a vector space?

1.2 Arrow Vectors and $\mathbb{R}^n$ for Small n

Many of you have seen vectors before, whether in physics, calculus, or perhaps your favorite animated movie. However, the definition you saw was perhaps a bit different. You likely learned that a vector is some quantity with both magnitude and direction, such as velocity. Well, does this match up with what we've said here about vector spaces? Indeed, it would be very embarrassing if it did not.[10]

10: Hey! What are all these pictures doing in our margin?

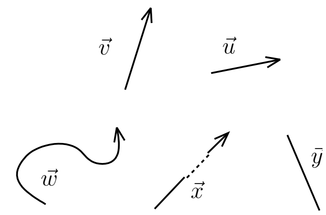


Figure 1.1: Here $\vec{v}$ and $\vec{u}$ are arrow vectors. However, $\vec{w}$, $\vec{x}$, and $\vec{y}$ are not.

If we let a vector be a quantity with magnitude and direction, it is natural to represent this with an arrow. The arrow points in the direction desired, and it has a length that can be used to represent its magnitude. For now, let's call this an **arrow vector**. See Figure 1.1 to the side.

Thus, we would like to establish that an arrow vector is actually an element in some vector space over $\mathbb{R}$, as we defined in the previous section. For these arrow vectors to form a vector space over $\mathbb{R}$ so that the set, V, we are dealing with is the set of all arrow vectors, we need to have a way to add them, and we need to have a concept of scalar multiplication with scalars from $\mathbb{R}$.

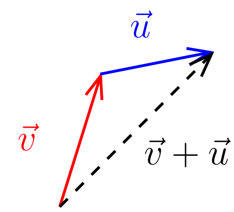


Figure 1.2: Here $\vec{v}$ and $\vec{u}$ are added to form the new arrow vector $\vec{v}+\vec{u}$, identified with the dashed arrow line.

First of all, since all these arrows are doing is recording magnitude and direction, their placement on this page does not matter. Thus, an arrow vector can be moved to a new location without changing the arrow vector itself. Thus, a natural addition of the vectors $\vec{v}$ and $\vec{u}$ is to first follow along $\vec{v}$ and then from there, follow along $\vec{u}$. As seen in Figure 1.2, the sum is then the arrow vector drawn from where you started to where you ended.

Now, does this addition satisfy our axioms for addition in a vector space? We see quickly that it is closed since the result is a new arrow vector. It is also associative and commutative from the diagrams in Figures 1.3 and 1.4 respectively. What about inverses and the zero vector? The inverse of a vector should just reverse direction or put the arrow on the other end as in Figure 1.5. The zero vector is pictured in Figure 1.6; it is very hard to see as it has magnitude 0. If you zoom in a lot, you might think that you'll be able to see it. However, even when you zoom in a lot, it still has magnitude 0, so it will be very hard (yes, impossible) to see. Some people like to use a dot for the zero vector, but we find our convention to be more accurate.

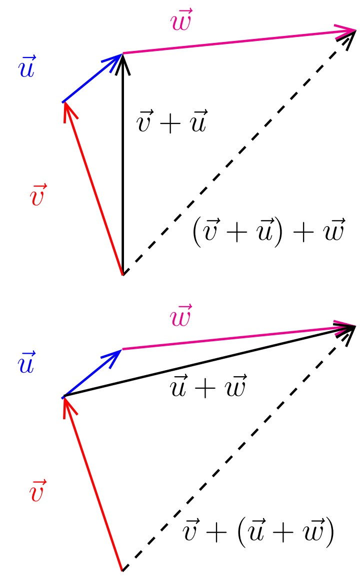


Figure 1.3: Here $\vec{v}$, $\vec{u}$, and $\vec{w}$ are added to form the new arrow vector $\vec{v}+\vec{u}+\vec{w}$, identified with the dashed arrow line, two ways to show this vector addition is associative.

What should scalar multiplication be? Well, we know it must satisfy repeated addition such as $2\vec{v} = \vec{v}+\vec{v}$. The right hand side of this gives an arrow vector that is twice as long as $\vec{v}$ but still in the same direction. Since our scalar multiplication must agree with this, we will define it to be a scaling of the length. Thus, $\alpha\vec{v}$ is an arrow vector in the direction of $\vec{v}$, but of length α times the length of $\vec{v}$ as in Figure 1.7. Convince yourself that this satisfies the rest of the axioms.

Connection to $\mathbb{R}^n$ (for small n).

Now that we've established that our arrow vectors actually form a vector space, how does this relate to $\mathbb{R}^n$ for small[11] n? Well, if we include the restriction that our arrow vectors must begin at the origin in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$, then it becomes fairly straightforward to show that this is equivalent to how we've already defined the vector spaces $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$!

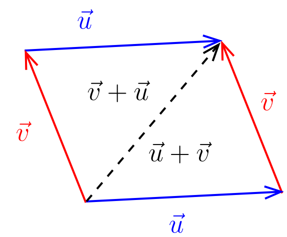


Figure 1.4: Here vector addition is shown to be commutative; this is sometimes, unimaginatively and with overstated importance, called the Parallelogram Law.

Let us try this with $\mathbb{R}^2$, for example. If the arrow vector $\vec{v}$ begins at the origin and extends to the point $(x, y) \in \mathbb{R}^2$, then we can call this the column vector

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Since all vectors begin at the origin, the tip of the arrow vector determines the vector itself, so this naturally defines a relation from the set of points in $\mathbb{R}^2$ to the set of arrows in the plane beginning at the origin. Similarly, one could define a relation from the set of arrows in the plane beginning at the origin to the set of points in $\mathbb{R}^2$. We'll get into the extent to which these sets are "the same" later, but for now, we strongly suspect you'll agree that these sets are similar enough to think of them interchangeably.

Figure 1.5: This is a vector $\vec{v}$ and its inverse $-\vec{v}$.

Exploration 14 Let's do an example here to see how these arrow vectors agree with $\mathbb{R}^2$.

- Draw the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as described above on the grid below. Include labels.

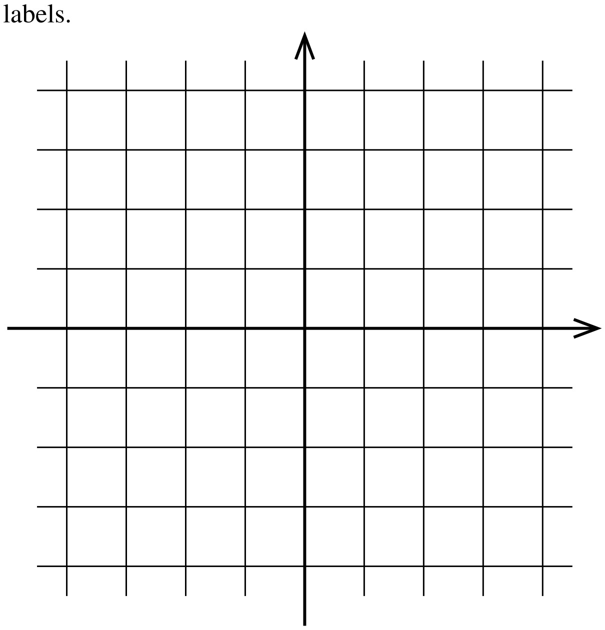


Figure 1.6: The zero vector is the arrow with no magnitude; it is pictured above. It is very hard to see as it has magnitude 0.

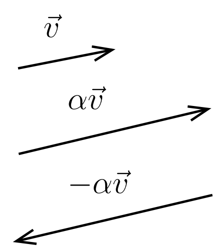


Figure 1.7: Here $\vec{v}$ is scalar multiplied by α and $-\alpha$ for some positive scalar α.

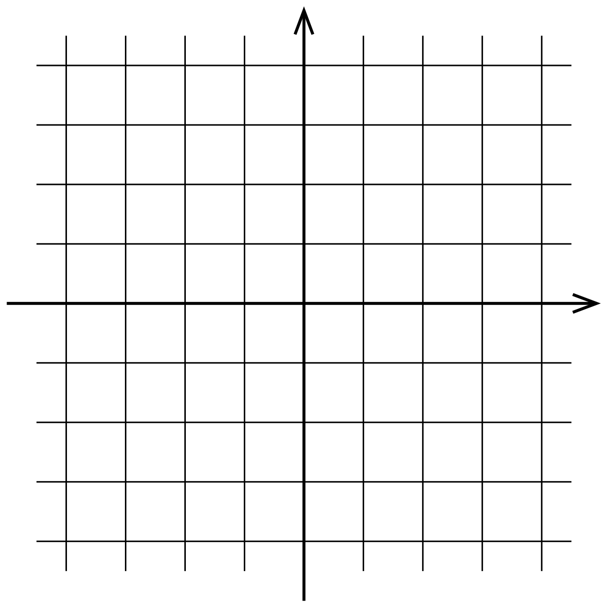
- Then, from the arrow tip of the vector you just drew, go up one square and to the left 2.
- Draw the arrow vector from the origin to the place you ended above. This is the sum of the vectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so you should have drawn the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Did you?

Exploration 15 We haven't mentioned scalar multiplication. Let's do an example with that one.

- Draw the vector $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ on the grid provided.

- Now, draw a line from the tip of the arrow down to the positive x-axis. This gives you a right triangle, and you can find the length of $\vec{v}$ using the Pythagorean theorem.[12]

- Now use the Pythagorean theorem again and compute the length of $2\vec{v} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. Did you get twice the length of $\vec{v}$?



In that last exploration, we used the Pythagorean theorem to find the length of our vectors when viewed as vectors in $\mathbb{R}^2$. Well, what about when they're in $\mathbb{R}^n$? Although we no longer have our handy arrow vectors for visualization in $\mathbb{R}^n$ for $n \geq 4$, we do actually have a way to discuss distances and lengths, so that some of the geometry that feels natural in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be extended to these other cases. The next section will set this up for us.

More Geometry with $\mathbb{R}^n$

As you'll recall, we gave up our notion of multiplication between two vectors in favor of scalar multiplication when we defined vector spaces in Definition 1.1.2. That doesn't stop people from trying to "multiply" two vectors anyway; there are a couple of different notions of "multiplication of vectors" out there. At least one of them ends up being pretty useful:

Definition 1.2.1 *The **inner product** is the function* $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *defined by relating two vectors to the real number given by summing the products of like components of the two vectors. That is, given* $\vec{v}, \vec{u} \in \mathbb{R}^n$*, we denote*

12: If you have forgotten what knowledge our good friend Pythagoras imparted, you should Google it. Then write an essay about how wonderful it is so that you never forget again.

Or maybe a tattoo would be better. . .

the inner product of $\vec{v}$ and $\vec{u}$ as $\vec{v} \cdot \vec{u}$, given by

$$\vec{v} \cdot \vec{u} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = v_1 u_1 + \cdots + v_n u_n = \sum_{i=1}^{n} v_i u_i.$$

Inner product is also synonymously called **scalar product**[13] and **dot product**.[14]

13: This was either named by codomain enthusiasts or Professor Ignacious J. Scalar whose surname is derived from the Spanish infinitive that translates roughly as "to cover with scales, weigh, and then climb".

14: This term was coined by. . . um. . . some one named Dot?

Do better, Ricky.

Exploration 16 Let's see this in action! Let

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \text{and } \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

be vectors in $\mathbb{R}^3$. Then $\vec{v} \cdot \vec{u} = 0 + 0 + 6 = 6$ and $\vec{u} \cdot \vec{w} = -2 + 0 + 12 = 10$. Find $\vec{v} \cdot \vec{w}$.

Exploration 17 Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Why doesn't $\vec{v} \cdot \vec{u}$ make sense?

Exploration 18 Is the inner product commutative? That is, for vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^n$, is it always true that $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$? Compute an example to illustrate your conclusion.

It turns out inner product has a lot of nice properties. Since it's used to define length, that's probably a good thing.

Theorem 1.2.1 *Let $\vec{v}$, $\vec{u}$, and $\vec{w}$ be vectors in the $\mathbb{R}^n$, and let a be a scalar. Then*

(a) $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$

(b) $(\vec{v} + \vec{u}) \cdot \vec{w} = \vec{v} \cdot \vec{w} + \vec{u} \cdot \vec{w}$

(c) $(a\vec{v}) \cdot \vec{u} = a(\vec{v} \cdot \vec{u}) = \vec{v} \cdot (a\vec{u})$

(d) $\vec{u} \cdot \vec{u} \geq 0$ *with* $\vec{u} \cdot \vec{u} = 0$ *if and only if* $\vec{u} = \vec{0}$.

Exploration 19 Let's walk through the proof of Theorem 1.2.1. We will need general forms of the vectors $\vec{v}$, $\vec{u}$, and $\vec{w}$ for this, so let

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \text{and } \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

(a) First, let's show that, as you suspected, the inner product is commutative.

$$\begin{aligned}\vec{v}\cdot\vec{u} =& v_1u_1 + \cdots + v_nu_n\\ =& u_1v_1 + \cdots + u_nv_n = \vec{u}\cdot\vec{v}\end{aligned}$$

(b) For this one, first, compute $(\vec{v}+\vec{u})\cdot\vec{w}$.

Now, compute $\vec{v}\cdot\vec{w}+\vec{u}\cdot\vec{w}$. (Then, they should be the same!)

(c) Note that $c\vec{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$. Compute $(c\vec{v})\cdot\vec{u}$.

Now, compute $c(\vec{v}\cdot\vec{u})$.

Lastly, compute $\vec{v}\cdot(c\vec{u})$.

(d) Note that $\vec{u}\cdot\vec{u} = u_1^2+\cdots+u_n^2$. Why must this always be nonnegative?

Now, for the last part, suppose $\vec{u}\cdot\vec{u} = u_1^2+\cdots+u_n^2 = 0$. Then, each of the u_i must be zero for $1 \leq i \leq n$. Thus, $\vec{u} = \vec{0}$. Also, if we compute $\vec{0}\cdot\vec{0}$, we see this must be 0.

Definition 1.2.2 ***Length*** *(or* ***norm****) is the function* $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$ *defined for any* $\vec{v} \in \mathbb{R}^n$ *as*

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2+\cdots+v_n^2}.$$

A vector $\vec{v} \in \mathbb{R}^n$ *is said to be a* ***unit vector*** *(or to have* ***unit length****) if* $\|\vec{v}\| = 1$.

If we're thinking of our vectors in $\mathbb{R}^n$ as having the two properties, magnitude and direction, then the inner product gives us a way to identify the magnitude (that is, length) of a vector. Unit vectors, having length 1, are a nice way to look at just the direction of a vector. The process of making a nonzero vector into a unit vector by dividing it by its length is sometimes called **normalizing a vector**.

Theorem 1.2.2 *For any nonzero vector* $\vec{v} \in \mathbb{R}^n$, $\dfrac{\vec{v}}{\|\vec{v}\|}$ *is a unit vector.*

PROOF.

$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \sqrt{\frac{\vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}} = \frac{\sqrt{\vec{v} \cdot \vec{v}}}{\sqrt{\|\vec{v}\|^2}} = \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1.$$

$\square$

Exploration 20 Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

- Find the length of the vector $\vec{v}$, denoted by $\|\vec{v}\|$.
- Find a vector with the same direction as $\vec{v}$ but with length 1.
- Find a vector with the same direction as $\vec{v}$ but with length 5.

We can think of distance between points in $\mathbb{R}^n$ (for $n \leq 3$). There are some formulae you may have seen:

- For $x, y \in \mathbb{R}$, the distance between x and y is
$$d(x, y) = \sqrt{(x-y)^2} = |x - y|,$$
which is just the usual absolute value in $\mathbb{R}$.
- For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$
This is sometimes called "the distance formula".
- For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we have
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Yeah, there's a pattern there. This is because these are all specific versions of the same formula. We state it below for vectors, rather than for points.

Definition 1.2.3 ***Distance*** *is the function* $dist\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *defined by relating two vectors to the length of their difference. That is, given* $\vec{v}, \vec{u} \in \mathbb{R}^n$*, we denote the distance between* $\vec{v}$ *and* $\vec{u}$ *as* $\operatorname{dist}(\vec{v}, \vec{u})$ *given by*

$$\operatorname{dist}(\vec{v}, \vec{u}) = \|\vec{v} - \vec{u}\|.$$

One of the immediate benefits of Definition 1.2.3 is that it works in $\mathbb{R}^n$ for larger[15] n. Indeed, it is difficult to imagine what distance looks like or means in $\mathbb{R}^7$. We invite you to try, just don't spend too much time trying. A better use of your time would be to make sure Definition 1.2.3 is really the same thing as our notion of distance between points in $\mathbb{R}^n$ for small n.

15: We define "larger" in this context to be an integer greater than 3.

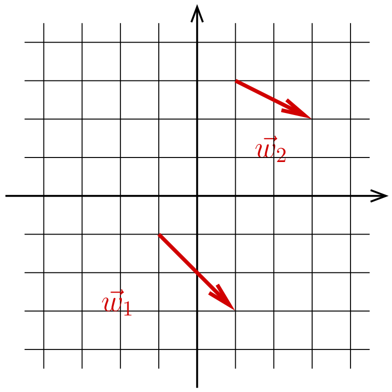
Section Highlights

The main idea of this section is to talk about some geometric properties for the real vector spaces $\mathbb{R}^n$.

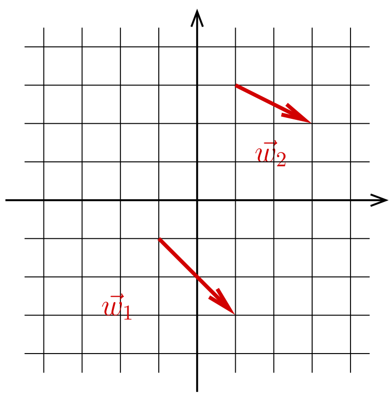
- ▶ A vector space can be formed from a set of arrows in either $\mathbb{R}^2$ or $\mathbb{R}^3$ using carefully chosen definitions for vector addition (Figure 1.2) and scalar multiplication (Figure 1.7).
- ▶ An arrow vector starting at the origin in $\mathbb{R}^2$ can be associated with the vector in $\mathbb{R}^2$ defined by the coordinates of the point at the tip of the arrow. The same can be done in $\mathbb{R}^3$. These can be used to view arrow vectors as a graphical representation of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- ▶ The length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the actual length of the associated arrow vector.
- ▶ The distance between two vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is the distance between the tips of the two arrow vectors starting at the origin.
- ▶ This geometry can be generalized to $\mathbb{R}^n$ with the help of an *inner product* (*dot product*). See Definition 1.2.1.
- ▶ The dot product allows us to extend geometric concepts like the length of a vector (Definition 1.2.2) and the distance between two vectors (Definition 1.2.3) to the vector space $\mathbb{R}^n$.

Exercises for Section 1.2

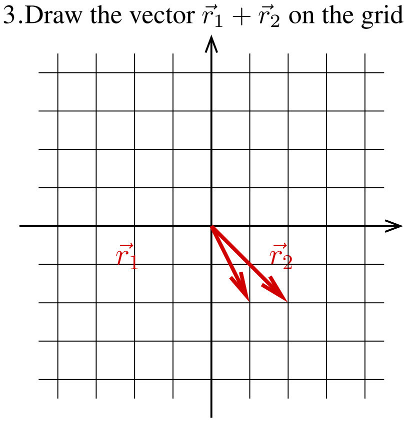
1.2.1. Draw the vector $\vec{w}_1 + \vec{w}_2$ on the grid below.



1.2.2. Draw the vector $\vec{w}_1 - \vec{w}_2$ on the grid below.

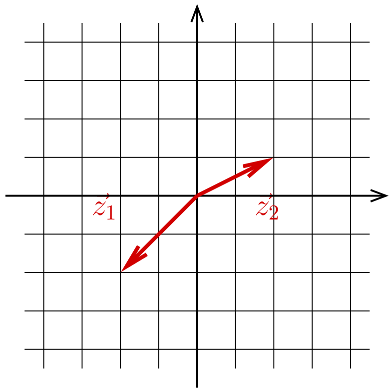


1.2.3. Draw the vector $\vec{r}_1 + \vec{r}_2$ on the grid below.



Then, find the column vector representations of $\vec{r}_1$ and $\vec{r}_2$ in $\mathbb{R}^2$. Use these to find $\vec{r}_1 + \vec{r}_2$. Does this agree with what you drew?

1.2.4. Draw the vector $\vec{z}_1 - \vec{z}_2$ on the grid below.



Then, find the column vector representations of $\vec{z}_1$ and $\vec{z}_2$ in $\mathbb{R}^2$. Use these to find $\vec{z}_1 - \vec{z}_2$. Does this agree with what you drew?

1.2.5. Let

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \qquad \text{and } \vec{c} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}.$$

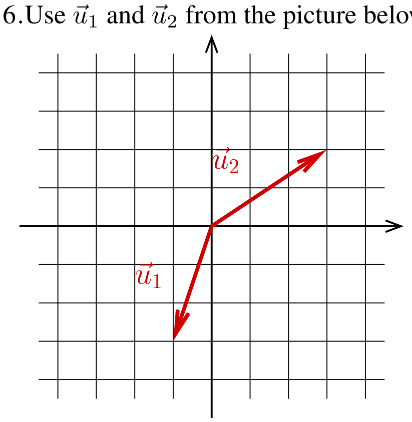
Note that $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^2$, which is a vector space. Simplify the following expressions down to a single vector, and indicate which properties of Definition 1.1.2 you use at each step.

(a) $3(\vec{a} - 2\vec{b}) + 2(\vec{b} + \vec{c})$

(b) $5(\vec{c} + 2\vec{b}) - 2(\vec{b} - 3\vec{a}) + 3(\vec{a} - 3\vec{b} - 2\vec{c})$

Sketch each term of each expression (for example, $3(\vec{a} - 2\vec{b})$ and $2(\vec{b} + \vec{c})$ in part a) on the same grid with the simplified vector.

1.2.6. Use $\vec{u}_1$ and $\vec{u}_2$ from the picture below to answer the questions.

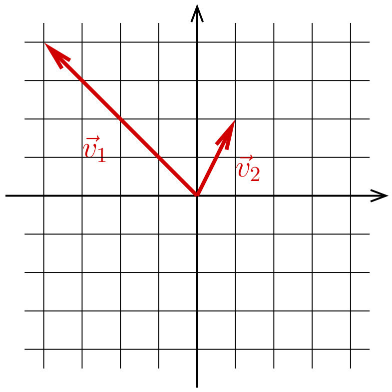


(a) Find $||\vec{u}_1||$ and $||\vec{u}_2||$.

(b) Find a unit vector in $\mathbb{R}^2$ in the direction of $\vec{u}_1$.

(c) Find a vector in $\mathbb{R}^2$ in the direction of $\vec{u}_1$ with length 7.

1.2.7. Use $\vec{v}_1$ and $\vec{v}_2$ from the picture below to answer the questions.



(a) Find $||\vec{v}_1||$ and $||\vec{v}_2||$.

(b) Find a unit vector in $\mathbb{R}^2$ in the direction of $\vec{v}_2$.

(c) Find a vector in $\mathbb{R}^2$ in the direction of $\vec{v}_2$ with length 2.

1.2.8. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find $\vec{v} \cdot \vec{u}$.

(b) Find $\vec{v} \cdot (2\vec{u})$.

(c) Find $(2\vec{v}) \cdot \vec{u}$.

(d) Find a nonzero vector $\vec{w}$ for which $\vec{v} \cdot \vec{w} = 0$.

1.2.9. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find $\vec{v} \cdot \vec{u}$.

(b) Find a nonzero vector $\vec{w}$ for which $\vec{v} \cdot \vec{w} = 0$.

1.2.10. Let's define a different product between vectors in $\mathbb{R}^3$. Let this product, denoted by $\boxtimes$, be given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \boxtimes \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 2v_1u_1 + 4v_2u_2 + 2v_3u_3.$$

Determine whether each property in Theorem 1.2.1 holds for $\boxtimes$.

1.2.11. Let's again define a different product between vectors in $\mathbb{R}^3$. Let this product, denoted by ☹, be given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \frownie \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = v_1u_3 + v_2u_2 + v_3u_1.$$

Determine whether each property in Theorem 1.2.1 holds for ☹.

1.3 Linear Independence and Span

Let's detour for a minute and talk about chess. A chessboard is an 8 by 8 grid, and there are several different pieces with rules about how they can move. For instance, the rook can move forward, backward, left or right, but not diagonally. The pawn can only move forward or diagonally forward if it is capturing another piece. Now, we come to the reason for our detour. Which spaces on the board can be reached by moving any piece using its specific rules? For the rook, we can reach any space on the board by moving up and over in the grid pattern. However, for the pawn, the spaces behind its starting space are unobtainable since it can only move forward.

Now, let's go back to vector spaces. Suppose we start at a vector $\vec{v}$ in a vector space V, and we are allowed to use our operations of scalar multiplication and vector addition to "move around" the vector space with $\vec{v}$. What other vectors can we obtain this way? What if we are only allowed to add certain other vectors from V? This sounds fun like chess, right?[16]

16: One of the authors finds this much more fun than chess.

Linear Combinations and Span

Since vector spaces are closed under vector addition and scalar multiplication, we can do both of these operations as many times as we want to vectors in a vector space V and still end up with a vector in V. This is so convenient that it gets its own name and definition:

Definition 1.3.1 *Let V be a vector space, $\vec{v}_1, \dots, \vec{v}_p \in V$, and $a_1, \dots, a_p \in \mathbb{R}$. The vector in V*

$$a_1\vec{v}_1 + \cdots + a_p\vec{v}_p \tag{1.8}$$

*is called a **linear combination** of the vectors $\vec{v}_1, \dots, \vec{v}_p$ with **weights** (or scalars) $a_1, \dots, a_p$.*

Exploration 21 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Now pick your two favorite real numbers. Did you pick 3 and 4? Great! Here's a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$3\vec{v}_1 + 4\vec{v}_2 = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 6+0 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}.$$

► Compute the linear combination $4\vec{v}_1 + 3\vec{v}_2$.

Suppose we are instead given a vector and asked whether or not it is a linear combination of some set of vectors.[17] How should this be handled? Well, we

17: This is just like our chess discussion! Which spaces can be reached with the allowed "moves?"

would need to find the appropriate scalars to make it a linear combination or show that no such scalars are possible. Let's see an example of this.

- Is $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

  Well, if it is, then there exist $a, b \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 = \vec{x}$. This means

  $$a \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

  Using the definitions of scalar multiplication and vector addition for $\mathbb{R}^3$, we can see that this means

  $$\begin{bmatrix} a+b \\ 2a \\ -a+b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

  This gives us equations $a + b = 3$, $2a = 4$ and $-a + b = -1$. These solve to get $a = 2$ and $b = 1$. Thus,

  $$2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix},$$

  which is true!

- Is $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

  Again, if it is, there exist $a, b \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 = \vec{y}$. This means

  $$a \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

  Just like before, this gives us equations. Here, they are $a + b = 1$, $2a = 1$ and $-a + b = 1$. However, unlike last time, these equations have no common solution; note that from the second equation, we have $a = 1/2$, which implies $b = 1/2$ from the first equation, but these values do not work in the third equation. Thus, we conclude that $\vec{y}$ is *not* a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

- Is $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Example 1.3.1 Now that we've seen examples in $\mathbb{R}^3$, let's see how things change in $\mathbb{P}_3$. Consider the vectors

$$\vec{p}_1 = 1 + x, \quad \vec{p}_2 = x, \quad \vec{p}_3 = x^3$$

in $\mathbb{P}_3$.

- Is $\vec{q} = 7 - x + 9x^3$ a linear combination of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$? Yes! To see that, we should solve for $a, b, c \in \mathbb{R}$ such that $a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 = \vec{q}$. That is,

  $$a(1 + x) + b(x) + c(x^3) = 7 - x + 9x^3.$$

  Rearranging the left hand side gives us

  $$a + (a + b)x + cx^3 = 7 - x + 9x^3.$$

  Solving for these scalars then gives us $a = 7$, $b = -8$, and $c = 9$. Then

  $$\vec{q} = 7 - x + 9x^3 = 7\vec{p}_1 - 8\vec{p}_2 + 9\vec{p}_3$$

  is a linear combination of the three vectors $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$.
- Now, let's consider

  $$\vec{v} = 7 - x + x^2 + 9x^3.$$

  Is $\vec{v}$ a linear combination of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$? Nope! No sum or rescaling of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ will produce the x^2 term in $\vec{v}$.

We've spent a bit of time now asking whether or not a vector is a linear combination of some set of vectors, so let's just formalize this a bit.

Definition 1.3.2 *Let V be a vector space and $\{\vec{v}_1, \dots, \vec{v}_p\} \subseteq V$. The **span of** $\vec{v}_1, \dots, \vec{v}_p$, denoted* $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$, *is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_p$. That is,*

(1.9)
$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_p\} = \{a_1\vec{v}_1 + \cdots + a_p\vec{v}_p \colon a_i \in \mathbb{R} \textit{ for } i = 1, \dots, p\}.$$

Exploration 22 Is $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ in $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$?
Hint: This is another way to ask a question we've already asked, so you should be able to answer it by looking back at a previous example.

Example 1.3.2 Let's revisit the vectors in Example 1.3.1. That is, consider again the vectors

$$\vec{p}_1 = 1 + x, \quad \vec{p}_2 = x, \quad \vec{p}_3 = x^3$$

in $\mathbb{P}_3$. Now, what is $\operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$? Well, let's figure it out.

$$\begin{aligned}\operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} &= \{a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 \colon a, b, c \in \mathbb{R}\} \\ &= \{a(1 + x) + b(x) + c(x^3) \colon a, b, c \in \mathbb{R}\} \\ &= \{a + (a + b)x + cx^3 \colon a, b, c \in \mathbb{R}\}.\end{aligned}$$

Note here that we have coefficients a, $a+b$, and c. There is obviously some relationship between a and $a+b$, but in this case, the relation doesn't really matter. Because b is not related to a or c and can be any real number, we could actually replace $a+b$ with a new variable $d = a+b$ that can be any real number. Then we get

$$\text{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \{a + dx + cx^3 : a, d, c \in \mathbb{R}\}.$$

This makes it clear that the span is all polynomials in $\mathbb{P}_3$ without an x^2 term.

We've seen in these explorations and examples how to determine whether a specific vector is or is not in the span of some set of vectors. Then, this last example gave us some idea about how to compute span algebraically. Now, let's talk a bit about the geometry and the bigger picture of what's in a span.

Example 1.3.3 First of all, what does the span of a single vector "look like?" Well, to picture anything, we should really think about the case of $\mathbb{R}^2$ or $\mathbb{R}^3$. Since $\mathbb{R}^2$ is much easier to draw, let's start there. Let's look at the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\text{Span}\{\vec{v}\}$ is just the set of scalar multiples of $\vec{v}$, which really forms the line that contains the vector $\vec{v}$. See Figure 1.8.

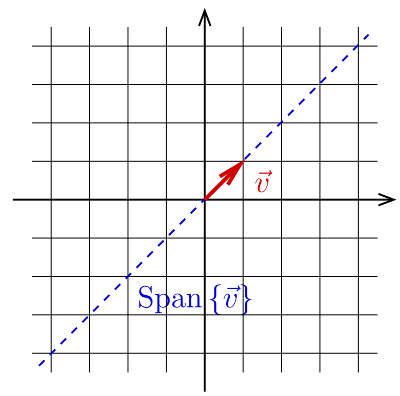


Figure 1.8: The single vector $\vec{v}$ is shown with a solid arrow line, and its span, $\text{Span}\{\vec{v}\}$ is shown with a dashed arrow line.

In the example above, note that we chose a nonzero vector $\vec{v}$. What's $\text{Span}\left\{\vec{0}\right\}$? Well, it's just $\vec{0}$ since any scalar multiple of $\vec{0}$ is just again $\vec{0}$. Now, let's see an example with two nonzero vectors in $\mathbb{R}^3$.

Example 1.3.4 Here are two vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$. Then

$$\begin{aligned} \text{Span}\{\vec{u}_1, \vec{u}_2\} &= \{a_1\vec{u}_1 + a_2\vec{u}_2 : a_i \in \mathbb{R}\} \\ &= \left\{ a_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} : a_i \in \mathbb{R} \right\} \end{aligned}$$

- First, note that a valid choice for a_2 is 0. Then $\text{Span}\{\vec{u}_1, \vec{u}_2\}$ contains any scalar multiple of $\vec{u}_1$, including $\vec{u}_1$ itself. Geometrically, this is the line containing $\vec{u}_1$. It's also $\text{Span}\{\vec{u}_1\}$! The same can be done for $\vec{u}_2$; refer to Figure 1.9.
- What should we expect for the span of two vectors then? Good question. Start with $\text{Span}\{\vec{u}_1\} = \{a_1\vec{u}_1 : a_1 \in \mathbb{R}\}$; to get $\text{Span}\{\vec{u}_1, \vec{u}_2\} = \{a_1\vec{u}_1 + a_2\vec{u}_2 : a_i \in \mathbb{R}\}$, we need to add any scalar multiple of $\vec{u}_2$ to any scalar multiple of $\vec{u}_1$. This means we're going to get $\text{Span}\{\vec{u}_2\}$ through any point of $\text{Span}\{\vec{u}_1\}$; in Figure 1.9, this is the red lines through any point on one of the blue lines, yielding the plane containing the two vectors $\vec{u}_1$ and $\vec{u}_2$.

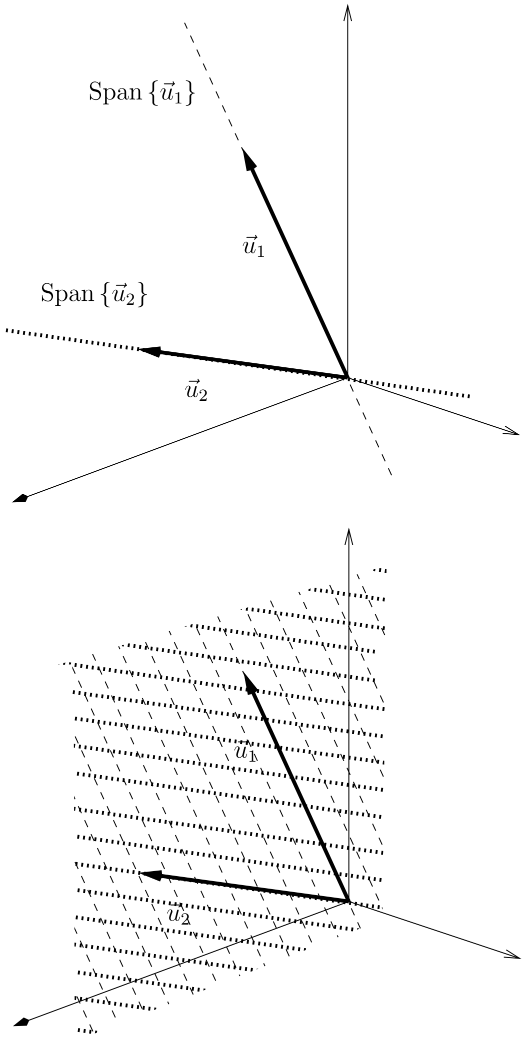


Figure 1.9: In the first image, Span $\{\vec{u}_1\}$ is the dashed line, and Span $\{\vec{u}_2\}$ is the dotted line; the second is Span $\{\vec{u}_1, \vec{u}_2\}$
.

Here's another fun question. Are either of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

in Span $\{\vec{u}_1, \vec{u}_2\}$? Nope. Note that any linear combination of $\vec{u}_1$ and $\vec{u}_2$ will have a zero in the second component. Neither of the given vectors have zero in the second component, so neither is a linear combination of $\vec{u}_1$ and $\vec{u}_2$. Thus, neither is in Span $\{\vec{u}_1, \vec{u}_2\}$.

Is the span of two vectors in $\mathbb{R}^3$ always a plane? Again, good question. If one of your vectors is the zero vector, then you just get the span of the other vector, so the answer is firmly "no." Ok. Fine. What about the span of two *nonzero* vectors in $\mathbb{R}^3$? Is that always a plane?

Example 1.3.5 Here are two vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

in $\mathbb{R}^3$. Note that $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$. Specifically, $2\vec{v}_1 = \vec{v}_2$. Watch what happens now to the span of these two vectors:

$$\begin{aligned}
\text{Span}\{\vec{v}_1, \vec{v}_2\} &= \{a_1\vec{v}_1 + a_2\vec{v}_2 \colon a_1, a_2 \in \mathbb{R}\} \\
&= \{a_1\vec{v}_1 + a_2(2\vec{v}_1) \colon a_1, a_2 \in \mathbb{R}\} \\
&= \{a_1\vec{v}_1 + 2a_2\vec{v}_1 \colon a_1, a_2 \in \mathbb{R}\} \\
&= \{(a_1 + 2a_2)\vec{v}_1 \colon a_1, a_2 \in \mathbb{R}\} \\
&= \{a\vec{v}_1 \colon a \in \mathbb{R}\} = \text{Span}\{\vec{v}_1\}.
\end{aligned}$$

Here, we used the fact that *any* $a \in \mathbb{R}$ can be realized as $a_1 + 2a_2$ for $a_1, a_2 \in \mathbb{R}$. Note that this is equivalent to showing $\{a_1 + 2a_2 : a_1, a_2 \in \mathbb{R}\}$ is equal to $\mathbb{R}$.
What does this mean for the span? Since $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$, there is redundancy in the span of the vectors. Thus, the span of these vectors will form a line, not a plane.

In the example above, the fact that one vector was a scalar multiple of the other gave us redundancy, so we were able to more efficiently write $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ as $\text{Span}\{\vec{v}_1\}$. We could also have written it as $\text{Span}\{\vec{v}_2\}$. When can we not do this? We definitely can't remove all the vectors, so is there a condition that says you can't remove a vector?

Linear Independence

Yes! This one! This one! Notions of dependence and independence between vectors can be used to detect the kind redundancy (or the lack of it) we saw in the previous example.

Definition 1.3.3 *A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is said to be **linearly independent** if*

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \tag{1.10}$$

*only when $a_1 = \cdots = a_n = 0$. The set $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is said to be **linearly dependent** if there are scalars $a_1, \ldots, a_n \in \mathbb{R}$, not all 0, such that*

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}. \tag{1.11}$$

Before we explore how this affects the span of a set of vectors, let's spend some time getting comfortable with the definition.

Example 1.3.6 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent or linearly dependent? To answer this, suppose there exist scalars $a, b, c \in \mathbb{R}$ such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$. If we can find nonzero a, b, and c, then we know they are dependent. If we cannot, then they are independent. Let's try to find them!

$$a\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This leads us to the equations $2a + b - c = 0$, $b + c = 0$, and $a = 0$. These equations have the unique solution $a = b = c = 0$, so the set is linearly independent.

Example 1.3.7 Let's replace $\vec{v}_3$ above with a new vector we'll call $\vec{v}_4$ and consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$$

Now, is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ linearly independent or linearly dependent? Suppose that $a, b, c \in \mathbb{R}$ are scalars such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_4 = \vec{0}$. That is,

$$a\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives us the equations $2a + b + 4c = 0$, $b + 2c = 0$, and $a + c = 0$. The latter two simplify to give $b = -2c$ and $a = -c$. Substituting these into the first equation gives us $0 = 0$, which is of course true but seemingly unhelpful. However, recall that if we get an equation like $0 = 0$, this often suggests that there are multiple solutions, and we can try to find one by choosing a value for one of the variables. If we choose $c = 1$, we will have $a = -1$ and $b = -2$. We can quickly check that this works to give us a solution to $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_4 = \vec{0}$ where the scalars are nonzero, and therefore, the set is linearly dependent. Note here that we could have made a different choice for c and found a different solution.

Exploration 23 Now it's your turn! Here are three vectors. Are they linearly independent or linearly dependent?

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Exploration 24 Let's explore the situation when we have a set with two vectors in it.

► Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, determine whether the following sets are linearly independent or linearly dependent. Hint: exactly two of these sets are linearly dependent.

(a) $\{\vec{v}_1, \vec{v}_2\}$

(b) $\{\vec{v}_1, \vec{v}_3\}$

(c) $\{\vec{v}_1, \vec{v}_4\}$

► Let V be a vector space and $\vec{u}, \vec{v} \in V$. Is there an easy way to tell if $\{\vec{u}, \vec{v}\}$ is a linearly independent set? In other words, is there an advantage to only dealing with two vectors when determining linear independence?

► Above, the vector $\vec{v}_4$ is the zero vector in $\mathbb{R}^3$. Can the zero vector ever be included in a linearly independent set?

Exploration 25 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

► We can write $\vec{v}_1$ as a linear combination of $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$. To do this, solve for $a, b, c \in \mathbb{R}$ so that

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- ► Now, rearrange to see that $\vec{v}_1 - a\vec{v}_2 - b\vec{v}_3 - c\vec{v}_4 = \vec{0}$. What does this tell us about the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$?

Did you say the vectors are linearly dependent? That is correct!![18] This works in general and gives us another way to think about linear dependence. . . Now, let's prove it!

18: Confetti!! Picture lots of colorful confetti!

Theorem 1.3.1 *A set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of two or more vectors, is linearly dependent if and only if one of the vectors is a linear combination of the other vectors.*

The "if and only if" bit means this is a biconditional statement. P if and only if Q means precisely the following two things: P implies Q *and* Q implies P.[19] This means that these conditions are logically equivalent. so we can use this as a test of dependence/independence. This is fairly common in mathematics; an "if and only if" is usually a vehicle for an alternative way of thinking about something.

19: Biconditional statements are great. It's a logical two-for-one deal. In one statement, you get two logical implications!

PROOF. Suppose $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a linearly dependent set. Then there are weights $a_1, \dots, a_p \in \mathbb{R}$ not all zero such that

$$a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \vec{0}.$$

We can assume without loss of generality[20] that it is the first weight that is nonzero, so $a_1 \neq 0$. Thus,

$$a_1\vec{v}_1 = -a_2\vec{v}_2 - \cdots - a_p\vec{v}_p.$$

Since $a_1 \neq 0$, we may multiply both sides of the equation by $1/a_1$, so

$$\vec{v}_1 = \frac{-a_2}{a_1}\vec{v}_2 + \cdots + \frac{-a_p}{a_1}\vec{v}_p.$$

Thus, v_1 is a linear combination of the other vectors.

20: "Without loss of generality" (sometimes abbreviated WLOG) means we are making a new, specific assumption that does effect the generality of the proof process. It often involves simply reordering or relabelling things.

Good point, RIcky, but did you notice why we were able to assume one of our scalars was nonzero?

Now suppose one of the vectors in the set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a linear combination of the other vectors. Again, we may assume without loss of generality that the guilty vector is the first one; that is, v_1 is a linear combination of the other vectors. Then

$$\vec{v}_1 = a_2\vec{v}_2 + \cdots + a_n\vec{v}_p.$$

We may rewrite this as

$$\vec{v}_1 - a_2\vec{v}_2 - \cdots - a_n\vec{v}_p = \vec{0}.$$

The weight on $\vec{v}_1$ is not zero, so by definition, $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a linearly dependent set. □

With this theorem in hand, we are ready now to talk about how linear independence affects our computations of the span of a set of vectors.

More Span

Let's start small, but not *too* small. Three vectors should be enough. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set of vectors in a vector space V. Then by Theorem 1.3.1, we know one of these vectors can be written as a linear combination of the others. For our purposes, it's okay to suppose that $\vec{v}_1$ is a linear combination of $\vec{v}_2$ and $\vec{v}_3$. That is,

$$\vec{v}_1 = b\vec{v}_2 + c\vec{v}_3$$

for some $b, c \in \mathbb{R}$.[21] Now, let's see how this connects to span.

21: It'll be clear in just a second why we skipped a...

$$\begin{aligned}
\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} &= \{a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 \colon a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{a_1(b\vec{v}_2 + c\vec{v}_3) + a_2\vec{v}_2 + a_3\vec{v}_3 \colon a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{(a_1 b + a_2)\vec{v}_2 + (a_1 c + a_3)\vec{v}_3 \colon a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{d_1\vec{v}_2 + d_2\vec{v}_3 \colon d_1, d_2 \in \mathbb{R}\} \\
&= \operatorname{Span}\{\vec{v}_2, \vec{v}_3\}.
\end{aligned}$$

Here, we use the fact that for any $a_1, b, c \in \mathbb{R}$, the sets $\{a_1 b + a_2 \colon a_2 \in \mathbb{R}\}$ and $\{a_1 c + a_3 \colon a_3 \in \mathbb{R}\}$ are both equal to $\mathbb{R}$.

Let's talk a bit about what we just did. We started with the span of three vectors, and we were able to reduce to a set of two vectors that has the same span as the original set. This is something we can do in general.

Theorem 1.3.2 *If S is a linearly dependent set of vectors in some vector space V, then there is some vector $\vec{v}$ in S such that* $\operatorname{Span}\{S\} = \operatorname{Span}\{S \backslash \{\vec{v}\}\}$.

The proof of this theorem is very similar to the discussion preceding it, so we'll leave the details to the exercises for now.

Exploration 26 Consider the following vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

- ▶ The set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find scalars $a, b, c, d \in \mathbb{R}$, not all 0, so that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 = \vec{0}$. Hint: There isn't a unique answer. You'll need to make a choice for one of the variables.

- ▶ From the equation above, it should be possible to identity a vector in S that could be removed without changing the span of the set of vectors. Actually, there are three vectors that could be chosen as the one removed! The only one that cannot be removed is $\vec{v}_4$. Why is this?

Example 1.3.8 Let's see how this looks in $\mathbb{P}_2$. Consider the vectors

$$\vec{p}_1 = 1 + x^2, \quad \vec{p}_2 = x, \quad \vec{p}_3 = 1 + 3x + x^2.$$

The set $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly dependent. To see this, we will find scalars $a, b, c \in \mathbb{R}$, not all zero, so that $a\vec{p}_1 + b\vec{p}_2 + c\vec{p}_3 = 0$. That is,

$$a(1 + x^2) + bx + c(1 + 3x + x^2) = 0.$$

Rearranging, we have

$$(a + c) + (b + 3c)x + (a + c)x^2 = 0.$$

Therefore $a + c = 0$ and $b + 3c = 0$. This gives us $a = -c$ and $b = -3c$. If we choose $c = 1$, we can see that $\vec{p}_3 = \vec{p}_1 + 3\vec{p}_2$. Now, we have realized $\vec{p}_3$ as a linear combination of $\vec{p}_1$ and $\vec{p}_2$, but we could rearrange this equation to realize $\vec{p}_1$ as a linear combination of $\vec{p}_2$ and $\vec{p}_3$ or $\vec{p}_2$ as a linear combination of $\vec{p}_1$ and $\vec{p}_3$. All of this tells us that $\operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \operatorname{Span}\{\vec{p}_1, \vec{p}_2\} = \operatorname{Span}\{\vec{p}_2, \vec{p}_3\} = \operatorname{Span}\{\vec{p}_1, \vec{p}_3\}$.

Let's talk about what Theorem 1.3.1 tells us about the span of a linearly independent set of vectors. Suppose S is a linearly independent set of vectors and $\vec{v} \in S$. By Theorem 1.3.1, we know $\vec{v}$ is *not* a linear combination of other vectors in S, since if it were, the set would be linearly dependent. Thus, if we were to compare $\operatorname{Span}\{S\}$ with $\operatorname{Span}\{S\backslash\{\vec{v}\}\}$, these would be different! In particular, $\vec{v} \in \operatorname{Span}\{S\}$, but $\vec{v} \notin \operatorname{Span}\{S\backslash\{\vec{v}\}\}$.

Section Highlights

- A *linear combination* of a set of vectors is a sum of scalar multiplied vectors from the set. See Definition 1.3.1.
- The set of all possible linear combinations of a set of vectors is the *span* of that set of vectors. See Definition 1.3.2.
- A system of equations can be set up and solved to determine whether a set of vectors is linearly independent. See Example 1.3.6 and Explorations 23 and 24.
- A set of vectors can be shown to be linearly dependent by finding one vector as a linear combination of the others. See Example 1.3.6 and Exploration 25.
- A system of equations can be set up and solved to determine whether the vector $\vec{v}$ in $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$. See Exploration 22.
- A linearly dependent set of vectors contains vectors that can be removed without altering the span of that set of vectors. The ones that can be removed are determined by dependence relations. See Exploration 26.

Exercises for Section 1.3

1.3.1. Some linear combinations are given, but they are missing some information. Fill in the missing information.

(a) In $\mathbb{R}^2$,
$$3\begin{bmatrix} \_\_\_\_ \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

(b) In $\mathbb{R}^2$,
$$3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

(c) In $\mathbb{R}^3$,
$$2\begin{bmatrix} 1 \\ \_\_\_\_ \\ -2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 2 \\ \_\_\_\_ \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}.$$

(d) In $\mathbb{P}_2$,
$$3(1+x) + 2(1-x-x^2) - (\_\_\_\_) = 5 - 2x - 3x^2.$$

1.3.2. Find the linear combination $5\vec{x} + 3\vec{y} - 2\vec{z}$ for the vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ given below.

(a) $\vec{x} = 3x + 2x^2$, $\vec{y} = 1 + x^2$, $\vec{z} = 3$ in $\mathbb{P}_2$

(b) $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$

(c) $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$

1.3.3. Determine whether $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is in each span below:

(a) $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} \right\}$

(c) $\operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}$

(d) $\operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(e) $\operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$

(f) $\operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(g) $\operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(h) $\operatorname{Span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

1.3.4. Consider the set $\text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$. Which of the vectors below are in this set?

(a) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$

(e) $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

(f) $\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$

(g) $\begin{bmatrix} -2 \\ 2 \\ a \end{bmatrix}$ for any $a \in \mathbb{R}$

(h) $\begin{bmatrix} b \\ -b \\ a \end{bmatrix}$ for any $a, b \in \mathbb{R}$

1.3.5. Let $\vec{u}$ and $\vec{v}$ be vectors in some vector space V. Explain why $\vec{u}$ and $\vec{v}$ are both vectors in $\text{Span}\,\{\vec{u}, \vec{v}\}$.

1.3.6. Suppose $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subset of vectors from a vector space V.

(a) Suppose $\vec{u} \in \text{Span}\,\{S\}$. Explain how this implies $\text{Span}\,\{\vec{u}\} \subseteq \text{Span}\,\{S\}$.

(b) Suppose $\vec{u}, \vec{v} \in \text{Span}\,\{S\}$. Explain how this implies $\text{Span}\,\{\vec{u}, \vec{v}\} \subseteq \text{Span}\,\{S\}$.

1.3.7. Let $\vec{u}$ and $\vec{v}$ be vectors in some vector space V.

(a) Explain why $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are in $\text{Span}\,\{\vec{u}, \vec{v}\}$.

(b) Show that $\vec{u}$ and $\vec{v}$ are both vectors in $\text{Span}\,\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$.

(c) What can you conclude then about $\text{Span}\,\{\vec{u}, \vec{v}\}$ and $\text{Span}\,\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$?

1.3.8. Let H be the set of all vectors in $\mathbb{R}^3$ of the form

$$\left\{\begin{bmatrix} -a_1 - 3a_2 \\ 4a_1 \\ a_1 - 2a_2 \end{bmatrix} : a_1, a_2 \in \mathbb{R}\right\}.$$

Rewrite this as a linear combination of two vectors with coefficients a_1 and a_2. Use this to find two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $H = \text{Span}\,\{\vec{v}_1, \vec{v}_2\}$.

1.3.9. Let K be the set of all vectors in $\mathbb{R}^3$ of the form

$$\left\{\begin{bmatrix} a_1 + a_2 + a_3 \\ 4a_1 \\ a_1 - 2a_2 - 2a_3 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

Rewrite this as a linear combination of three vectors with coefficients a_1, a_2 and a_3. Then find two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $K = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

1.3.10. Let J be the set of all vectors in $\mathbb{R}^3$ of the form

$$\left\{ \begin{bmatrix} -a_1 - 3a_2 \\ 4a_3 \\ a_1 - 2a_2 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Find vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ such that $J = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

1.3.11. Determine whether the sets below are linearly independent or linearly dependent.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(g) $\{1 + x, x + x^2, x^2\}$

(h) $\{2 + 2x, 1 - x + x^2, 4 + 2x^2\}$

(i) $\{1 + x + x^2, x + x^2, x^2\}$

(j) $\{1 + x, 2x, 1 - x^2, 1 + x^2\}$

(k) $\{1 + x, 2x + x^2, x^2, 1\}$

1.3.12. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set in a vector space V. Which of the following sets must also be linearly independent? Give a complete argument to support your conclusion.

(a) $\{\vec{v}_2, \vec{v}_3, \vec{v}_1\}$

(b) $\{\vec{v}_1, \vec{v}_3\}$

(c) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_2\}$

(d) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$

(e) $\{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, \vec{v}_1 - \vec{v}_3\}$

(f) $\{\vec{v}_2 - \vec{v}_3, \vec{v}_1 - \vec{v}_2\}$

1.3.13. The set of vectors below is linearly dependent. However, it contains many linearly independent subsets. Find all nonempty linearly independent subsets. There should be 11.

$$\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Circle the ones that have the same span as the original set. (Hint: They should all be the same size.)

1.3.14. For each set S below, reduce the set to a linearly independent one that has the same span.

(a) $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \right\}$

(b) $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

(c) $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

In Problems 1.3.15, 1.3.16, and 1.3.17 below, determine what the span of the vectors looks like geometrically. Explicitly, find whether it is a point, a line, a plane, or all of $\mathbb{R}^3$.

1.3.15. Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Determine what $\operatorname{Span}\{\vec{v}, \vec{u}\}$ looks like geometrically.

1.3.16. Let $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. Determine what $\operatorname{Span}\{\vec{w}, \vec{z}\}$ looks like geometrically.

1.3.17. Let $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. Determine what $\operatorname{Span}\{\vec{u}, \vec{w}, \vec{z}\}$ looks like geometrically.

1.3.18. Consider the following vectors in $\mathbb{P}_2$:

$$\begin{aligned} \vec{p}_1 &= \pi + \pi^2 x + \pi^3 x^2 \\ \vec{p}_2 &= \pi - x^2 \\ \vec{p}_3 &= \pi^2 - \pi x^2. \end{aligned}$$

Show that $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is a linearly dependent set.

1.3.19. Let

$$\begin{aligned} \vec{p}_1 &= x + x^3 \\ \vec{p}_2 &= 1 + x^2 \\ \vec{p}_3 &= 1 + x \end{aligned}$$

be vectors in $\mathbb{P}_4$. Describe $\operatorname{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ algebraically (in set notation) and in words.

1.3.20. Let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^2$. Then $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ for some $a, b \in \mathbb{R}$. Find a way to write $\vec{w}$ as a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Explain why $\mathbb{R}^2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

1.3.21. Show that $\mathbb{R}^2 = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right\}$. (Hint: Follow the technique of the previous exercise.)

1.3.22. Let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^3$. Then $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$. Find a way to write $\vec{w}$ as a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Conclude that $\mathbb{R}^3 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

1.3.23. Let $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $J = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Show that $H = J$.

1.3.24. Let $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $J = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Show that $H \neq J$.

1.4 Subspaces

Vector spaces are sometimes too big. Oftentimes, the collection of vectors you actually care about is only a small piece of the vector space with which you are stuck. In some cases, a collection of vectors in a vector space turns out to be a vector space itself, and this can be very convenient. Who needs extraneous vectors just hanging around everywhere? You know what's *not* convenient? Verifying all ten axioms to show that the smaller set of vectors is a vector space.[22]

Ready for some good news?

Definition 1.4.1 *A* ***subspace*** *of a vector space V is a subset H of V with the following three properties:*

- *The zero vector is in H.*
- *(Closure under vector addition) For any $\vec{v}$ and $\vec{u}$ in H, the vector $\vec{v}+\vec{u}$ is also in H.*
- *(Closure under scalar multiplication) For any $\vec{v}$ in H and any a in $\mathbb{R}$, the vector $a\vec{v}$ is also in H.*

Note that vector additional and scalar multiplication for H are the same as for V.

It only makes sense that a sub*space* is a vector *space*. Thus, if you want to show a set is a vector space and it's actually a subset of some vector space, then you can just use this definition instead to show it is a subspace, right? That is *definitely* better than verifying all those axioms in the definition of a vector space. It really seems too good to be true, though, right? Well, we were due for some good news; it is true:

Theorem 1.4.1 *A subspace of a vector space is itself a vector space.*

PROOF. The three axioms for a subspace take care of three of our vector space axioms. Then the addition and scalar multiplication are the same as for the ambient vector space, and all the other properties are inherited. □

"Inherited" is an interesting math word, and it works a lot like one might expect. If H is a subset of a vector space V with the same operations as V, then properties of V are oftentimes also passed on to H, like from parent to offspring. For example, if V has properties such as commutativity and associativity for some operations, then provided H is closed under the operations, they still hold for H because the operations on H are the same as those on V.

Example 1.4.1 The set

$$H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \colon v_1 + v_2 + v_3 = 0 \right\}.$$

22: I don't like verifying all ten axioms. It takes too long.

Verifying axioms is a wonderful exercise that helps build understanding.

It turns out H is a subspace of $\mathbb{R}^3$, so H is a vector space. You can check this later, but for now, just trust us that it's true. Then

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in H \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \notin H.$$

This is because H is the set of vectors in $\mathbb{R}^3$ whose components sum to zero, and we have that $1 + (-1) + 0 = 0$ but $1 + (-1) + 1 \neq 0$. Thus, we can also check, more generally, that

$$H_0 = \left\{ \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} \in \mathbb{R}^3 \colon a \in \mathbb{R} \right\} \quad \subseteq \quad H, \quad \text{and}$$

$$\left\{ \begin{bmatrix} a \\ -a \\ 1 \end{bmatrix} \in \mathbb{R}^3 \colon a \in \mathbb{R} \right\} \quad \not\subseteq \quad H.$$

Since H_0 is a subset of the vector space H with the same operations as H, we would now only need to check the three axioms from the definition of subspace to verify that H_0 is also a vector space. It inherits the remaining vector space properties from H!

Now, before doing anything interesting, we should note that any vector space V has two uninteresting subspaces. It's not hard to check[23] that V is a subspace of itself. How does one check? Just verify each of the axioms[24] in Definition 1.4.1 for V. The other uninteresting subspace is the zero vector. Yep, this is the only finite set we'll get in this course that turns out to be a vector space. It's also a subspace of *any* vector space. You should check that, too, with Definition 1.4.1. A reasonable response to the question "How many subspaces does vector space V have?" is "At least two."[25] It turns out, however, that there are often[26] many more.

23: You should check.

24: You really should do this, too.

25: Unless $V = \{\vec{0}\}$.

26: When $V = \mathbb{R}$, there are exactly these two. Maybe we should show this, though. Exercise!

You mean like running laps?

No, at the end of the section. A homework exercise!

Example 1.4.2 Let's start off with the familiar vector space $\mathbb{P}_2$. Consider all polynomials of the form $a + bx$ for any $a, b \in \mathbb{R}$. Let's call this set H for convenience. H is a subset of $\mathbb{P}_2$ since $\mathbb{P}_2$ is all polynomials of the form $a + bx + cx^2$ for $a, b, c \in \mathbb{R}$, and we could let $c = 0$. Also, it is a subspace of $\mathbb{P}_2$:

- ▶ H contains the zero vector 0 since we could let $a = b = 0$.
- ▶ Let $a_1 + b_1x$ and $a_2 + b_2x$ be any polynomials in H. Then

  $$(a_1 + b_1x) + (a_2 + b_2x) = (a_1 + a_2) + (b_1 + b_2)x \in H$$

  since $(a_1 + a_2)$ and $(b_1 + b_2)$ are again in $\mathbb{R}$. Thus, H is closed under addition.
- ▶ Let $k \in \mathbb{R}$. Then $k(a + bx) = ka + kbx \in H$ since $ka, kb \in \mathbb{R}$, so H is closed under scalar multiplication.

Exploration 27 Now it's your turn! Let $J = \{a + cx^2 \colon a, c \in \mathbb{R}\}$. We see that these will all be either degree 2 or 0 polynomials, so $J \subseteq \mathbb{P}_2$. Show that all 3 of the axioms for a subspace are also satisfied.

- ▶ What should a and c be to see that J contains the zero vector?

- Let $a_1 + c_1x^2$ and $a_2 + c_2x^2$ be any polynomials in J. Show the sum of these two polynomials is still in J.

- Let $k \in \mathbb{R}$. Show that $k(a + cx^2) \in J$ for any $a + cx^2 \in J$.

Let's consider the two subspaces H and J of $\mathbb{P}_2$ for a bit. How many of you noticed that H is really $\mathbb{P}_1$? How often does something like this happen? Well, for $k \leq n$, we can always show that $\mathbb{P}_k$ is a subspace of $\mathbb{P}_n$.[27] However, these are not all of the subspaces of $\mathbb{P}_n$! The subspace J from the exploration above is not $\mathbb{P}_k$ for any integer k, and you'll see several other examples of subspaces of $\mathbb{P}_n$ in the exercises.

27: Expect to see this as an exercise!

We should really see an example of something that is *not* a subspace, too.

Example 1.4.3 Let's consider the set $K = \left\{ \begin{bmatrix} a \\ a+2 \end{bmatrix} : a \in \mathbb{R} \right\}$. Since for any $a \in \mathbb{R}$ we know $a + 2 \in \mathbb{R}$ as well, this is a subset of $\mathbb{R}^2$. However, it is not a subspace.

- First of all, this set does not contain the zero vector of $\mathbb{R}^2$. To see this, suppose for some $a \in \mathbb{R}^2$
$$\begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Then $a = 0$ and $a + 2 = 0$. Both of these can't be true. At this point, we can definitively state it is not a subspace, but let's see what happens with the other axioms.[28]

- We can check that closure under vector addition also fails. To show this, it is enough to check that it fails for specific vectors rather than general ones. Let's consider
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$
These are in K since one corresponds to $a = 0$ and the other to $a = 1$, but the sum
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
is not in K since we cannot have $a = 1$ and $a + 2 = 5$ both be true.

- Lastly, we can show that K is not closed under scalar multiplication. Again, to show a property fails, it is enough to give a specific case where it fails. We can choose
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \in K$$

28: We only need to fail one axiom?

Yep! Once you fail one axiom, you can't satisfy all axioms of a definition.

Neat.

and the scalar $5 \in \mathbb{R}$. Then

$$5\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

is not in K since $a = 0$ and $a + 2 = 10$ cannot both be true.

Exploration 28 The following subset L is not a subspace of $\mathbb{P}_1$. Which of the three axioms fail?[29]

$$L = \{mx + b \colon m, b > 0\}$$

29: Remember that we only need to show *one* axiom fails to show L is not a subspace!

Now that we've seen some examples, let's explore in detail what happens in $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$

If you've been keeping up with the sidenotes, then we've already mentioned the subspaces of $\mathbb{R}$. We've claimed that the only subspaces of $\mathbb{R}$ are the "uninteresting" ones, $\{\vec{0}\}$ and $\mathbb{R}$ itself. However, we also saw in the exercises of Section 1.1 that the interval $(0, \infty)$ is a vector space when addition is given by $a \boxplus b = ab$ and scalar multiplication is $ka = a^k$ for $k \in \mathbb{R}$ and $a, b \in (0, \infty)$.[30] Why is this not a subspace of $\mathbb{R}$? Well, the operations of addition and scalar multiplication *must* be the exact same ones from the larger vector space in order for a subset to be a subspace. Thus, Theorem 1.4.1 lets us say "A subspace of a vector space V is a subset that is also a vector space" but only with the caveat that the operations are the same as those for V.

30: Unless you skipped that one like I did.

Now let's focus on subspaces of $\mathbb{R}^3$. This will give us some nice geometric intuition to go along with the algebraic computations.

Example 1.4.4 First we'll show that

$$H = \left\{ \begin{bmatrix} 3t \\ 0 \\ -5t \end{bmatrix} : t \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$. Note that the vectors here have three real components, so H is a subset of $\mathbb{R}^3$. Next we need to verify the three axioms in Definition 1.4.1. The first requires that $\vec{0} \in H$. Indeed, when $t = 0$, we see that $\vec{0} \in H$. That one was easy (this time). It only remains to show H is closed under vector addition and scalar multiplication. As we did in Section 1.1, we need to have general elements of the set H to satisfy Definition 1.4.1, which needs to hold "for any" vectors in H. Let $x, y \in \mathbb{R}$, so

$$\vec{x} = \begin{bmatrix} 3x \\ 0 \\ -5x \end{bmatrix}, \vec{y} = \begin{bmatrix} 3y \\ 0 \\ -5y \end{bmatrix} \in H$$

are general elements. Let's see what happens when we add them:

$$\vec{x}+\vec{y}=\begin{bmatrix} 3x \\ 0 \\ -5x \end{bmatrix}+\begin{bmatrix} 3y \\ 0 \\ -5y \end{bmatrix}=\begin{bmatrix} 3x+3y \\ 0+0 \\ -5x-5y \end{bmatrix}=\begin{bmatrix} 3(x+y) \\ 0 \\ -5(x+y) \end{bmatrix}.$$

This last vector is an element of H since $x+y \in \mathbb{R}$. Thus, the sum of any two vectors in H is still in H, so H is closed under vector addition. To check closure under scalar multiplication, let $a \in \mathbb{R}$. Then

$$a\vec{x}=a\begin{bmatrix} 3x \\ 0 \\ -5x \end{bmatrix}=\begin{bmatrix} a(3x) \\ a(0) \\ a(-5x) \end{bmatrix}=\begin{bmatrix} 3(ax) \\ 0 \\ -5(ax) \end{bmatrix}.$$

Since $ax \in \mathbb{R}$, this last vector is also in H. Thus, H is closed under scalar multiplication. It follows that H is a subspace of $\mathbb{R}^3$.
Before moving on, let's think a bit about the geometry of this subspace in $\mathbb{R}^3$. Sketch this set of vectors in $\mathbb{R}^3$. Observe that H is actually a line through the origin in the direction of the vector

$$\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}.$$

What if we considered a line that does not go through the origin? Can that be a subspace of $\mathbb{R}^3$? This would be a fun exploration, but let's do this one together.

Example 1.4.5 Here's another subset of $\mathbb{R}^3$.

$$H=\left\{\begin{bmatrix} 3t \\ 1 \\ -5t \end{bmatrix}: t \in \mathbb{R}\right\}$$

The only difference between this set and the set in Example 1.4.4 is that these vectors have 1 as their second component, rather than 0. Geometrically, this is equivalent to a line in $\mathbb{R}^3$ that does not go through the origin, like we were just speculating about. It turns out this is not a subspace since it does not contain the zero vector from $\mathbb{R}^3$.

Example 1.4.6 At the risk of developing a theme, here's another subset of $\mathbb{R}^3$.

$$H=\left\{\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}: y, z \in \mathbb{R}\right\}$$

Spoiler: this one's a subspace. Checking is very similar to Example 1.4.4, but you should check it anyway. Let's consider

$$\left\{\begin{bmatrix} y \\ z \end{bmatrix}: y, z \in \mathbb{R}\right\}=\mathbb{R}^2.$$

Well, this seems like it's the same as H, in which case we'd have $\mathbb{R}^2$ as a subspace of $\mathbb{R}^3$ (just like $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$). Seems okay, right? ...Right?

No! Absolutely not! While they have many things in common and "look alike," the definition of a subspace H of the vector space V requires first for H to be a *subset* of V. Vectors in $\mathbb{R}^2$ are not vectors in $\mathbb{R}^3$; they are two different mathematical objects. While H resembles $\mathbb{R}^2$ in many ways, $\mathbb{R}^2$ *is not a subspace of* $\mathbb{R}^3$ because it's not even a subset of $\mathbb{R}^3$. You may find this annoying. Indeed, the vectors

$$\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \text{ and } \begin{bmatrix} y \\ z \end{bmatrix}$$

carry the same information. They are alike in many ways, but they are, strictly speaking, different mathematical objects. While this may seem obnoxious, this degree of rigor in definitions is necessary for consistently functional (and understandable) mathematics. Later we shall investigate what we can gain by defining and understanding what it means for vectors or vector spaces to "look alike."

Exploration 29 True or false? $\mathbb{R}$ is a subspace of $\mathbb{R}^3$.[31]

31: FALSE!!

What's different about the definition of $\mathbb{P}_n$ that allows $\mathbb{P}_k$ to be a subspace of $\mathbb{P}_n$ for $k \leq n$?

We've now seen examples of several subspaces of $\mathbb{R}^3$ and some subsets that are not subspaces in $\mathbb{R}^3$. What do subspaces of $\mathbb{R}^3$ look like geometrically?

- Well, we have $\{\vec{0}\}$ as a subspace. This is just the point at the origin.
- From Examples 1.4.4 and 1.4.5 we saw a subspace that was a line through the origin and that lines not traveling through the origin are not subspaces. Really, we saw one example of this, but they all fail for the exact same reason. Any line not through the origin does not contain $\vec{0}$.
- Then in Example 1.4.6, we saw that a plane could be a subspace. Are all planes subspaces, though? Nope. Just like lines must go through the origin, planes that are subspaces must also go through the origin in order to contain the zero vector.
- Lastly, we know that $\mathbb{R}^3$ is a subspace of itself, and we all know what $\mathbb{R}^3$ looks like, right?[32]

32: If you said a giant purple gorilla, you are incorrect.

We've discussed subspaces of $\mathbb{R}^n$ for $n = 1$ and $n = 3$. Note that we skipped over $\mathbb{R}^2$. That's because it will make a glorious exercise for you! We also have not addressed subspaces for $n > 3$. That's because once we leave the comfortable 3-dimensional world we live in, we lose our geometric tools and must rely upon just algebra. With that in mind, we will now turn to more algebraic tools.

Spans as Subspaces

Suppose you were given a set and you suspected it was a vector space. Initially, there were roughly ten axioms you had to verify before you could declare your set a vector space. At the beginning of this section, we cut that list down to three things if your set was already contained in a vector space. What's better than checking three things? One. Checking just one thing is better.[33]

33: Wait. What's the catch? There has to be a catch. This feels like a trap!

Theorem 1.4.2 *Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in a vector space V. Then* $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ *is a subspace of V. Note that* $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ *can be referred to as* ***the subspace spanned by*** $\vec{v}_1, \ldots, \vec{v}_n$.

PROOF. Recall that

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \{a_1\vec{v}_1 + \cdots + a_n\vec{v}_n \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}.$$

Since the a_i's can be any real number, we observe that when $a_1 = \cdots = a_n = 0$, we have

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0\vec{v}_1 + \cdots + 0\vec{v}_n = \vec{0},$$

so $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ contains the zero vector. Now we need general vectors in $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$:

$$\begin{aligned}\vec{x} &= a_1\vec{v}_1 + \cdots + a_n\vec{v}_n \\ \vec{y} &= b_1\vec{v}_1 + \cdots + b_n\vec{v}_n.\end{aligned}$$

To see closure under vector addition, we add $\vec{x}$ and $\vec{y}$:

$$\begin{aligned}\vec{x} + \vec{y} &= (a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) + (b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) \\ &= (a_1\vec{v}_1 + b_1\vec{v}_1) + \cdots + (a_n\vec{v}_n + b_n\vec{v}_n) \\ &= (a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n,\end{aligned}$$

noting that the only reason we can pull off all that algebraic manipulation (associativity and commutivity of vector addition and distributivity for scalar multiplication) is because these vectors are all part of a vector space V already. The last line of this equation is a vector in $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ since $a_i + b_i \in \mathbb{R}$, so $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is closed under vector addition. To see closure under scalar multiplication, we multiply $\vec{x}$ by $c \in \mathbb{R}$:

$$\begin{aligned}c\vec{x} &= c(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \\ &= ca_1\vec{v}_1 + \cdots + ca_n\vec{v}_n \\ &= (ca_1)\vec{v}_1 + \cdots + (ca_n)\vec{v}_n.\end{aligned}$$

Again, the last line of this equation is a vector in $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ since $ca_i \in \mathbb{R}$, so $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is closed under scalar multiplication. Behold! $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subspace of V! $\square$

How does one use this theorem? If you have a set of vectors in a vector space V and want to show they form a subspace, all you have to do is show your set is the span of some set of vectors. Just show that, and you're done. Pretty great, right?

Example 1.4.7 Let

$$H=\left\{\begin{bmatrix} a-2b \\ b-a \\ a \\ b \end{bmatrix} : a,b\in\mathbb{R}\right\}\subset\mathbb{R}^4.$$

You could show this is a subspace using the definition of a subspace, or you could show it's a subspace by showing it's the span of some set of vectors. Let's do the latter. Note that for any $a,b\in\mathbb{R}$,

$$\begin{bmatrix} a-2b \\ b-a \\ a \\ b \end{bmatrix}=\begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix}+\begin{bmatrix} -2b \\ b \\ 0 \\ b \end{bmatrix}=a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}+b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Neat, eh? Here we've *undone* vector addition and scalar multiplication, but look what we can do now:

$$\begin{aligned} H &= \left\{\begin{bmatrix} a-2b \\ b-a \\ a \\ b \end{bmatrix} : a,b\in\mathbb{R}\right\} \\ &= \left\{a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}+b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} : a,b\in\mathbb{R}\right\} \\ &= \operatorname{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

This tells us H is the span of two vectors in $\mathbb{R}^4$. Now it follows from Theorem 1.4.2 that H is a subspace of $\mathbb{R}^4$.

Exploration 30 Let

$$J=\left\{\begin{bmatrix} a+b \\ a+b+c \\ a+b \end{bmatrix} : a,b,c\in\mathbb{R}\right\}\subset\mathbb{R}^3.$$

Find a set of vectors $\{\vec{v}_1,\ldots,\vec{v}_k\}$ such that $J=\operatorname{Span}\{\vec{v}_1,\ldots,\vec{v}_k\}$.

We now have a theorem that says that the span of a set of vectors in a vector space V must be a subspace of V, which is neat. What about the other way around? Is every subspace of V a span of some set of vectors in V? Actually, this turns out to be true! It's very exciting. Unfortunately, the proof of this fact requires a bit more than the scope of this text, so this is really all we'll say about it.

Intersections and Sums of Subspaces

Perhaps you have two subspaces of a particular vector space V that you are interested in. A natural question would perhaps be "How are they related to one another?" or maybe instead "How could you combine these subspaces?" The first of these two questions leads us to the idea of the intersection of two subspaces. First, let's be sure we know what an intersection is.

Definition 1.4.2 *The intersection of two sets A and B is all of the elements that are in both A and B. We denote this intersection as $A \cap B$.*

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the intersection of A and B is $\{2, 4\}$.

Now, what is the intersection of two subspaces of a vector space V? Why, a subspace of V!

Theorem 1.4.3 *Let V be a vector space over $\mathbb{R}$ with subspaces U and W. Then the intersection $U \cap W$ is also a subspace of V.*

Exploration 31 Let's go through this proof together.

- We need to argue that $\vec{0}$ is in $U \cap W$. Thus, we need $\vec{0}$ to be in *both* U and W. Why is this true?

- Now, let $\vec{x}$ and $\vec{y}$ be in $U \cap W$. Why is $\vec{x} + \vec{y}$ in U? Why is it in W?

  Since $\vec{x} + \vec{y}$ is in both U and W, it must be in $U \cap W$. Thus the intersection is closed under addition.

- Lastly, suppose $k \in \mathbb{R}$ and $\vec{x}$ is again in $U \cap W$. Follow the logic above to show $k\vec{x}$ must be in $U \cap W$.

Exploration 32 Let's look at an explicit example of this. The following are all subspaces of $\mathbb{R}^3$.

$$U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let's find the intersections of these subspaces.

- $U \cap V$: A vector that is in both U and V will satisfy the following equation

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{1.12}$$

for some real numbers a, b, and c. The top row (first component) gives us the equation $a = 0$. Then, the middle row (second component) gives us the equation $a = c$. The last row (third component) gives us $b = 0$. Thus, $a = b = c = 0$. So the only vector satisfying Equation 1.12 is the zero vector and $U \cap V = \{\vec{0}\}$.

- $U \cap W$: A vector that is in both U and W will satisfy the following equation

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{1.13}$$

for some real numbers a, b, c and d. Find the equations from each row and show that $a = b = c = d$.

Now, what does that tell us about the intersection? Well, any vector in the intersection must be of the form

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{or equivalently} \quad a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, $U \cap W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

- $V \cap W$: Follow the methods used above to compute this intersection.

The second question mentioned earlier dealt with combining subspaces. The correct notion for this is to take the *sum* of the subspaces.

Definition 1.4.3 *Let U and W be subspaces of a vector space V. The* ***sum*** *of these subspaces $U + W$ is defined as*

$$\{\vec{u} + \vec{w} \colon \vec{u} \in U, \vec{w} \in W\}.$$

Additionally, if U and W have the property that $U \cap W = \{\vec{0}\}$, then we call this a ***direct sum*** *and denote it $U \oplus W$.*

Theorem 1.4.4 *Let U and W be subspaces of a vector space V. Then the sum $U + W$ is a subspace of V.*

The proof of this follows a similar format as the previous one that said $U \cap V$ is a subspace, so we'll just leave that as an exercise.[34]

34: At this point, you should have a good idea of what the homework for this section will look like!

Example 1.4.8 Let's determine the sums of some of the vector spaces we considered in Exploration 31. Let's recall the definitions of the subspaces of $\mathbb{R}^3$ to start.

$$U = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \quad V = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

$$W = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

Whenever our subspaces are given to us as the span of a set of vectors, finding the sum is fairly straightforward. We just combine the sets defining each of the two subspaces to get a new set, then take the span. For example,

$$\begin{aligned} U + W &= \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \\ &= \text{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \mathbb{R}^3 \end{aligned}$$

Note here that the initial list of vectors we obtained was not linearly independent, despite the fact the original two lists were when considered separately. This will always be the case when your subspaces have an intersection other than $\{\vec{0}\}$.

Let's consider now the case of $U + V$. Since we know from Exploration 31 that $U \cap V = \{\vec{0}\}$, what we'll have instead is the direct sum $U \oplus V$. In particular,

$$U \oplus V = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \mathbb{R}^3.$$

Note that here the new list of vectors was linearly independent.

Direct sums give us a way to decompose a vector space neatly and will be a topic that comes up again in the following chapters.

Section Highlights

- A *subspace* (Definition 1.4.1) is a subset of a vector space that is itself a vector space.
- For a vector space V, V and the set consisting of just the zero vector are both subspaces.
- The span of a set of vectors is always a subspace. See Theorem 1.4.2.
- The sum of two subspaces is a subspace (Definition 1.4.3 and Theorem 1.4.4), and the intersection of two subspaces is a subspace (Theorem 1.4.3).

Exercises for Section 1.4

1.4.1. The following subsets all fail to be closed under vector addition. Give an example that illustrates this failure.

(a) $\{a + bx\colon a, b \in \mathbb{R}, a \neq 0\} \subset \mathbb{P}_2$

(b) $\left\{\begin{bmatrix} 1 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

(c) $\left\{\begin{bmatrix} ab \\ b \\ a \end{bmatrix} : a, b \in \mathbb{R}\right\} \subset \mathbb{R}^3$

(d) $\left\{\begin{bmatrix} a+3 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

1.4.2. The following subsets all fail to be closed under scalar multiplication. Give an example that illustrates this failure.

(a) $\{a + ax : a \in \mathbb{R}, a \geq 0\} \subset \mathbb{P}_2$

(b) $\left\{\begin{bmatrix} 1 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

(c) $\left\{\begin{bmatrix} ab \\ b \\ a \end{bmatrix} : a, b \in \mathbb{R}\right\} \subset \mathbb{R}^3$

(d) $\left\{\begin{bmatrix} a+3 \\ a \end{bmatrix} : a \in \mathbb{R}\right\} \subset \mathbb{R}^2$

1.4.3. Show that the following subsets are subspaces of $\mathbb{P}_3$. Then, write each as the span of some set of vectors.

(a) $\{a + bx^3\colon a, b \in \mathbb{R}\}$

(b) $\{4(a + c) + bx + cx^2\colon a, b, c \in \mathbb{R}\}$

1.4.4. Show that $\{4 + ax + bx^2\colon a, b \in \mathbb{R}\}$ is *not* a subspace of $\mathbb{P}_3$. Identify which of the three properties fail.

1.4.5. Show that

$$H = \left\{\begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R}\right\}$$

is a subspace of $\mathbb{R}^3$, and write it as the span of some set of vectors.

1.4.6. Show that

$$H = \left\{\begin{bmatrix} 0 \\ x \\ y \end{bmatrix} : x, y \in \mathbb{R}\right\}$$

is a subspace of $\mathbb{R}^3$, and write it as the span of some set of vectors.

1.4.7. Show that

$$H = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R} \right\}$$

is *not* a subspace of $\mathbb{R}^3$. Identify which of the three properties fail.

1.4.8. Show that

$$H = \left\{ \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R}, t > 0 \right\}$$

is *not* a subspace of $\mathbb{R}^3$. Identify which of the three properties fail.

1.4.9. Show that

$$H = \left\{ \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} : t \in \mathbb{R}, t \geq 0 \right\}$$

is *not* a subspace of $\mathbb{R}^3$. Identify which of the three properties fail.

1.4.10. Suppose V is a vector space. Show that $\{\vec{0}\}$ is a subspace of V.

1.4.11. Suppose V is a vector space. Show that V is a subspace of V.

1.4.12. Show that the only subspaces of $\mathbb{R}$ are $\mathbb{R}$ and $\{\vec{0}\}$. To do this, suppose there is some other subspace H of $\mathbb{R}$. If $H \neq \{\vec{0}\}$, then there must be some nonzero vector $v \in H$. Conclude that $H = \mathbb{R}$.

1.4.13. Suppose that H is a subset of a vector space V and you've shown that the second and third axioms from the definition of subspace hold (that is, that H is vector addition under and scalar multiplication). Did you know that if H is nonempty that this implies that the first axiom holds (that is, that H contains the zero vector) as well? It's true! Now prove it.

1.4.14. We will now investigate subspace in $\mathbb{R}^2$.

(a) Show $\{\vec{0}\}$ is a subspace of $\mathbb{R}^2$.

(b) Show $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

(c) Show the set $L(a, b)$ below is a subspace of $\mathbb{R}^2$ for any real numbers a and b.

$$L(a, b) = \left\{ \begin{bmatrix} ka \\ kb \end{bmatrix} : k \in \mathbb{R} \right\}.$$

1.4.15. Show that $\mathbb{P}_k = \{a_0 + a_1x + a_2x^2 + \cdots + a_kx^k : a_i \in \mathbb{R}$ for $0 \leq i \leq k\}$ is a subspace of $\mathbb{P}_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{R}$ for $0 \leq i \leq n\}$ for any $0 \leq k \leq n$. Hint: This should look similar to Example 1.4.2.

1.4.16. The following are all subspaces of $\mathbb{R}^3$.

$$U = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad V = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) Is $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in U \cap V$?

(b) Is $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in U \cap W$?

(c) Find $U \cap V$.

(d) Find $U \cap W$.

(e) Find $V \cap W$.

(f) Find $U + W$. Is this a direct sum?

(g) Find $V + W$. Is this a direct sum?

1.4.17. The following are all subspaces of $\mathbb{R}^3$.

$$U = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad V = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad W = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) Is $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in U \cap V$?

(b) Is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in U \cap W$?

(c) Is $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in V \cap W$?

(d) Find $U \cap V$.

(e) Find $U \cap W$.

(f) Find $V \cap W$.

(g) Find $U + W$. Is this a direct sum?

(h) Find $V + W$. Is this a direct sum?

1.4.18. Prove Theorem 1.4.4, which says the sum of two subspaces is a subspace.

1.4.19. Along with the concept of intersection, we often discuss the *union* of two sets. Let A and B be sets. The union of A and B, denoted $A \cup B$, is the set of all elements in either A or B. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the union of A and B is $\{1, 2, 3, 4, 6, 8\}$. Let U and W be subspaces of a vector space V. Show that $U \cup W$ is *not* a subspace of V in general. Under which conditions will it be a subspace?

1.5 A Menagerie of Vector Spaces

We've had quite a few different examples of real vectors spaces already. Here's a list with links to where you can go back and read about them again:

- $\mathbb{R}^n$: Equation 1.2, Section 1.1
- $\mathbb{P}_n$: Equation 1.7, Section 1.1
- arrows: beginning of Section 1.2
- $\mathbb{C}$: Exercise 1.1.3, Section 1.1
- $(0, \infty)$: Exercise 1.1.12, Section 1.1
- sums and intersections of subspaces: Theorem 1.4.4, Section 1.4

But wait! There's more![35]

35: I'm not paying for anything.

A Peek Into the Future

There are other examples of vector spaces that will appear for us naturally later. However, we can tell you a little about them now.[36]

36: Spoilers!

Example 1.5.1 A rectangular array of numbers with m rows and n columns is called an $m \times n$ matrix, and we call the set of all such matrices $\mathcal{M}_{m\times n}(\mathbb{R})$. When $m = 2$ and $n = 2$, we more specifically have

$$\mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

If we define vector addition componentwise,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+g & d+h \end{bmatrix},$$

and for scalars k, we define scalar multiplication as

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$$

You might have noticed that this set and its operations look a lot like $\mathbb{R}^4$, just with the vector entries arranged in a different shape. If that's the case, it should not surprise you that $\mathcal{M}_{2\times 2}(\mathbb{R})$ is a vector space. Should you require a review of the ten axioms, though, you should check them for practice.

Exploration 33 Let

$$\begin{aligned} M_0 &= = \left\{ \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{R} \right\} \text{ and} \\ M_1 &= = \left\{ \begin{bmatrix} 1 & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}. \end{aligned}$$

Show that M_0 is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$ but M_1 is not.

Example 1.5.2 Consider the general form of a linear equation in n variables:

$$a_1x_1 + \cdots + a_nx_n = b.$$

Moving the constant term to the left side, we have

$$a_1x_1 + \cdots + a_nx_n - b = 0,$$

and this looks very much like the general form of a vector in $\mathbb{P}_n$. Leveraging what we know about $\mathbb{P}_n$, we can make a vector space of linear equations. Let

$$V = \{a_1x_1 + \cdots + a_nx_n = a_{n+1} \colon a_i \in \mathbb{R} \text{ for } i = 1, \ldots, n+1\},$$

and define vector addition by combining like terms and scalar multiplication by multiplication on both sides of the equation. With these operations, this set is a vector space.

Example 1.5.3 Suppose V and W are both vector spaces. Just like how we took Cartesian products of sets in Chapter 0, we can do the same with vector spaces. Consider the set

$$V \times W = \{(\vec{v}, \vec{w}) \colon \vec{v} \in V \text{ and } \vec{w} \in W\}.$$

If we use operations from V in the first component and operations from W in the second, then the set $V \times W$ is a vector space. Check!

Exploration 34 Do these computations in $\mathbb{R}^2 \times \mathbb{P}_2$.

- $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 1 + x + x^2\right) + \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 3 + x^2\right) =$

- $5\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 1 + x + x^2\right) =$

Function Spaces

While we're on the topic of additional examples of vector spaces, here's a nice class of them that you may find interesting.[37]

Let $I \subseteq \mathbb{R}$ be any interval, and define

$$\begin{aligned} \mathcal{C}(I) &= \{f \colon I \to \mathbb{R} \colon f \text{ is continuous}\} \\ \mathcal{D}(I) &= \{f \colon I \to \mathbb{R} \colon f \text{ is differentiable}\} \\ \mathcal{R}(I) &= \{f \colon I \to \mathbb{R} \colon f \text{ is integrable}\} \end{aligned}$$

Again, we're going to think of $\vec{f} \in \mathcal{D}(I)$ as a vector, knowing in our hearts that this vector is a real-valued function defined on I. For $\vec{f}, \vec{g} \in \mathcal{C}(I)$ (or $\mathcal{D}(I)$ or $\mathcal{R}(I)$[38]) and $a \in \mathbb{R}$, define

$$\begin{aligned} \vec{f} + \vec{g} &= f \vec{+} g \colon I \to \mathbb{R} \text{ defined by } (f \vec{+} g)(x) = f(x) + g(x) \text{ and} \\ a\vec{f} &= a\vec{f} \colon I \to \mathbb{R} \text{ defined by } a\vec{f}(x) = af(x). \end{aligned}$$

37: Ooh! Look at all the fancy calligraphy!

C'mon, Ricky. Don't get taken in by substance-free flash; seriously–Wait. That "C" is very cool.

38: Hang on. Why "R" for "integrable?" Why not "I?"

Do you want to write "$I(I)$?" The "R" is probably for Riemann.

Using these typical operations, function addition and scalar multiplication of functions, $\mathcal{C}(I)$, $\mathcal{D}(I)$, and $\mathcal{R}(I)$ are all vector spaces. Of course, you should really check this. Let's talk about what would go into doing that. There are some properties of continuous, differentiable, and integrable[39] functions that should be easily located in just about any calculus text that tell us these sets are closed under the operations of vector addition and scalar multiplication. Then the remaining properties actually hold more broadly for the set of functions from I to $\mathbb{R}$.

39: I didn't come here for calculus.

Settle down. No one's asking you to integrate anything.

Example 1.5.4 Note that $\vec{f} = \sin x$ and $\vec{g} = \cos x$ are both vectors in $\mathcal{C}([0,1])$ because both $\sin x$ and $\cos x$ are continuous functions of x on the interval $[0,1]$. Note that $\sin \pi/4 = \cos \pi/4$. Nevertheless, $\vec{f} \neq \vec{g}$; for these vectors to be equal, they would have to be the same *for all values of* x in $[0,1]$.

Moreover, $\vec{f}$ and $\vec{g}$ are linearly independent. If $\vec{f}$ and $\vec{g}$ were linearly dependent, we would have $\vec{g} = a\vec{f}$ for some nonzero scalar a, and this would have to be true *for all* x. However, note that for $x = 0 \in [0,1]$, we have

$$1 = \cos 0 = a \sin 0 = 0,$$

so there is no a such that $\vec{g} = a\vec{f}$ for all $x \in [0,1]$. Thus, $\vec{f}$ and $\vec{g}$ are linearly independent.

Exploration 35 Let $\vec{f} = e^x$ and $\vec{g} = x^2$. We know that $\operatorname{Span}\left\{\vec{f}, \vec{g}\right\}$ is a subspace of $\mathcal{D}(\mathbb{R})$. Show that $\operatorname{Span}\left\{\vec{f}, \vec{g}\right\} \neq \mathcal{D}(\mathbb{R})$.

Example 1.5.5 Here's a differential equation:

$$y''' + 3y'' + 2y' = 0. \tag{1.14}$$

We can check that $y_1 = e^{-x}$, $y_2 = e^{-2x}$, and $y_3 = 87$ are all solutions:

$$\begin{aligned} y_1''' + 3y_1'' + 2y_1' &= -e^{-x} + 3e^{-x} - 2e^{-x} = 0 \\ y_2''' + 3y_2'' + 2y_2' &= -8e^{-x} + 12e^{-x} - 4e^{-x} = 0 \\ y_3''' + 3y_3'' + 2y_3' &= -0 + 3(0) - 2(0) = 0 \end{aligned}$$

Each function y_i for $i = 1, 2, 3$ makes Equation 1.14 true when substituted in for y, so all three are solutions for the differential equation.

This all may seem like a wild tangent, but note that y_1, y_2, and y_3 from Example 1.5.5 are all vectors in $\mathcal{D}(\mathbb{R})$. Obviously, $\operatorname{Span}\{y_1, y_2, y_3\}$ is a subspace of $\mathcal{D}(\mathbb{R})$.[40]

40: Wait, why is this obvious?

Exploration 36 Show that any vector in $\operatorname{Span}\{y_1, y_2, y_3\}$, that is, any linear combination of y_1, y_2, and y_3, is a solution to Equation 1.14.

Thus, $\mathrm{Span}\,\{y_1, y_2, y_3\}$ is an entire *subspace of solutions* for the given differential equation. This is an example of the Superposition Principle, and it actually holds for a large class of differential equations. We should definitely think about this more, but let's do it later.[41]

41: Later, like in a different book?

Exercises for Section 1.5

1.5.1.Let

$$H \quad = \quad = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } a + d = 0 \right\}.$$

Show H is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

1.5.2.Determine whether $H = \left\{ \begin{bmatrix} x & x+y \\ x-y & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of $\mathcal{M}_{2\times 2}$.

1.5.3.Consider the vector space

$$V = \{a_1x_1 + a_2x_2 = a_3 \colon a_i \in \mathbb{R} \text{ for } i = 1, 2, 3\}.$$

Note that $x_1 + x_2 = 2$ is a vector in V and that $x_1 = x_2 = 1$ is a solution to this particular equation. Determine whether the set of vectors in V for which $x_1 = x_2 = 1$ is a solution is a subspace of V. Justify your determination.

1.5.4.Let

$$S = \{(s_1, s_2, \dots) \colon s_i \in \mathbb{R} \text{ for } i = 1, 2, \dots\}$$

be the set of infinite real-valued sequences. Using

$$\begin{aligned} (s_1, s_2, \dots) + (r_1, r_2, \dots) &= (s_1 + r_2, s_2 + r_2, \dots) \text{ and} \\ k(s_1, s_2, \dots) &= (ks_1, ks_2, \dots), \end{aligned}$$

determine whether S is a vector space.

1.5.5.Show that $\mathbb{R}^2 \times \mathbb{P}_2$ is a vector space.

1.5.6.Let

$$S_0 = \{(s_1, s_2, \dots) \colon s_i \in \{0, 1\} \text{ for } i = 1, 2, \dots\}$$

be the set of infinite binary sequences. Here's how addition will work on $\{0, 1\}$:

$$\begin{aligned} 1 + 0 &= 1, \\ 0 + 1 &= 1, \\ 0 + 0 &= 0, \text{ and} \\ 1 + 1 &= 0. \end{aligned}$$

For $(s_1, s_2, \dots), (r_1, r_2, \dots) \in S_0$, and $k \in \{0, 1\}$, define

$$\begin{aligned} (s_1, s_2, \dots) + (r_1, r_2, \dots) &= (s_1 + r_2, s_2 + r_2, \dots) \text{ and} \\ k(s_1, s_2, \dots) &= (ks_1, ks_2, \dots). \end{aligned}$$

Determine whether S_0 is a vector space. Note that k is only allowed to be 0 or 1; we're not looking for a *real* vector space here because we're not using real scalars.

1.5.7.For two functions $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$, both differentiable, the Wronskian of f and g is defined as $W(x) = f(x)g'(x) - g(x)f'(x)$. Show that if $W(x) \neq 0$ for some x, then f and g are linearly independent.