4 More Fun with Matrices

4.1 Systems of Equations and Matrices

In the previous chapter, we introduced the concept of a matrix and explained the connections they enjoy with linear transformations. Many texts actually begin with matrices because they are a rich and convenient computational tool, especially useful for solving systems of equations. You'll note that many of the questions we asked previously boiled down to solving a system of equations. It shouldn't be a surprise then to learn that we can revisit topics, such as linear independence and coordinate vectors, using matrices. In this chapter, we will do all this. . . and more.

Systems of Equations Algebraically

As mentioned above, matrices are an important tool for solving systems of equations. Let's start there.

Consider this situation. A new dorm is being built on campus and the rooms come in two types. The first type is a "pod" which holds 6 students, and the second type is a standard room that holds 2 students. The university would like the dorm to hold 952 students in a total of 200 rooms.[1] We can set up a *system of equations* to represent this scenario and determine how many of each rooms should be included in the new dorm.

1: We're not sure why these specific numbers, but we know not to question such decisions.

Let x represent the number of pods and y represent the number of standard rooms. Then we have

$$6x + 2y = 952 \quad \text{and} \quad x + y = 200.$$

There are multiple ways you learned in previous algebra courses to solve this system of equations. You could graph these lines and see where they intersect.[2] You could solve one equation for x and then substitute the result into the other. Then there's *elimination*. We're going to use something pretty close to elimination to solve this by manipulating our system using two operations.

2: Do *not* do this. This is the worst way to solve this problem!

- We can replace either equation in the system by a scalar multiple of that equation.
- We can replace either equation in the system by the addition of that equation and a scalar multiple of the other equation.

Let's begin.

$$6x + 2y = 952 \tag{4.1}$$

$$x + y = 200 \tag{4.2}$$

We replace Equation 4.1 with Equation 4.1 scaled by $1/2$.

$$3x + y = 476 \tag{4.3}$$

$$x + y = 200 \tag{4.4}$$

We replace Equation 4.3 with Equation 4.3 plus Equation 4.4 scaled by (-1).

$$2x \quad = 276 \tag{4.5}$$

$$x + y = 200 \tag{4.6}$$

We scale Equation 4.5 by $1/2$.

$$x \quad = 138 \tag{4.7}$$

$$x + y = 200 \tag{4.8}$$

We replace Equation 4.8 by Equation 4.8 plus Equation 4.7 scaled by (-1).

$$x \quad = 138 \tag{4.9}$$

$$y = 62 \tag{4.10}$$

Solved! We need 138 pods and 62 standard rooms. Ready for some good news? We can use matrices as a shorthand for those operations. We'll use a matrix for the coefficients on the left hand side of our linear equations, a vertical bar to separate the left hand side from the right hand side, and a column for the constants on the right hand side. For example, given a system of m equations in n variables, such as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

we can form the matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

This matrix is called an **augmented matrix**, and there's definitely more to it than we're presenting here. For now, though, it certainly provides a more convenient notational convention. You're welcome.[3]

3: Hey! I'll handle the snarkiness around here!

Here's the previous example as an augmented matrix and the sequence of operations we did to it.

$$\left[\begin{array}{cc|c} 6 & 2 & 952 \\ 1 & 1 & 200 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 476 \\ 1 & 1 & 200 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 276 \\ 1 & 1 & 200 \end{array}\right] \to$$

$$\left[\begin{array}{cc|c} 1 & 0 & 138 \\ 1 & 1 & 200 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 138 \\ 0 & 1 & 62 \end{array}\right]$$

That's so much less to write! Matrices might be another way to solve systems of equations that is worth further exploration, right?[4]

4: You know what that means…

Exploration 103 Let's translate the rules we followed to solve the system of equations into the language of matrices.

- Any row may be replaced by a scalar multiple of itself.
- Any row may be replaced by the sum of that row and a scalar multiple of another row.

Suppose the total number of rooms in the above example were changed to 240. Solve this new system using the matrix notation.

Our primary concern in this section will be identifying solutions for systems of equations, so we should probably define *solution* formally.

Definition 4.1.1 *A* ***solution for a system*** *of linear equations*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

is an n-tuple $(x_1, \dots, x_n)$ that makes all the linear equations in the system true.

System Representations

The matrix notation we used at the beginning of the section was sold as a notational convenience, but it's actually a lot more natural than it may seem. Given a system of m equations in n variables,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array} \tag{4.11}$$

we could use the definition of the product of a scalar and a vector to write this as the **vector equation**

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}, \tag{4.12}$$

where for $1 \leq j \leq n$,

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Using the definition of the product of a matrix and a vector, we find that the vector equation (4.12) is equivalent to

$$[\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]\, \vec{x} = \vec{b},$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

This is a **matrix equation**

(4.13) $$A\vec{x} = \vec{b},$$

where

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

is called the **coefficient matrix**. We summarize this in the following theorem.

Theorem 4.1.1 *Using the matrix and vectors*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}. \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \textit{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

the following are all equivalent representations

(a) *System of equations:*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

(b) *Vector equation:*

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b},$$

(c) *Matrix equation:*

$$A\vec{x} = \vec{b},$$

(d) *Augmented matrix:*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

This theorem gives us the flexibility to view a system of equations in four different contexts.

Exploration 104 Consider the system below.

$$\begin{aligned} 6x_1 + 2x_2 - x_3 &= 111 \\ x_1 + 7x_2 - 2x_3 &= 56 \\ -5x_1 + x_2 + 4x_3 &= 12 \end{aligned}$$

Write it as a vector equation, matrix equation, and an augmented matrix. When you're done, do it again, but make it really, really small.

Gauss-Jordan Elimination

The goal here is to change our system of equations *without altering the solution(s)* to make it easier to identify the solution(s). Thinking back to the example at the beginning of this section (or other methods you've used to solve systems), this is precisely what one typically does. The strategy we are about to show you is called Gauss-Jordan elimination, named for Carl Friedrich Gauss and Wilhelm Jordan, despite the fact that Gauss apparently had nothing to do with it. First, we'll outline the operations we can do in this elimination procedure.[5] Next, we'll formalize a goal for the procedure so that we know when we can stop. Lastly, we'll see the procedure described as an algorithm. Got all that? Good. Let's get started.

5:  Two of these already made an appearance in that opening example.

Operations: Three Things You Can Do

What are things you can *do* to a system that don't change the solution for the system? Some things are so simple, they're not obvious. We could change the order of our equations; that certainly wouldn't alter any of the solutions for any of the equations, so the system solution would remain unchanged as well. We could also multiply both sides of an equation by a nonzero scalar; again, the solution set for that equation wouldn't change, so the system solution remains unchanged.

We keep "doing stuff" to equations; let's get a little more specific. Let V be the set of all linear equations in n variables with real coefficients. That is,

$$V = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\colon a_i, b \in \mathbb{R} \text{ for } 1 \le i \le n\}. \quad (4.14)$$

For any nonzero scalar α and any equation $v \in V$, define the nonzero scalar multiple of v by α as the equation one gets by multiplying both sides of v by α. For any $v, u \in V$, define the sum of the equations v and u to be the equation one gets by equating the sum of the left sides of v and u to the sum of the right sides of v and u. [6]

There is a third, less obvious, thing one could do to a system without changing the solution. Let's pile it onto the other two and call it a theorem.[7]

Theorem 4.1.2 *The following three operations on a system of m equations in n variables do not change the system's solution:*

(a) *interchanging the position of any two equations in the system;*
(b) *multiplying an equation by a nonzero scalar; and*
(c) *replacing the ith equation with the sum of the ith equation and any nonzero scalar multiple of any of the other equations.*

PROOF. We've already covered both a and b. To prove c, the following observation will simplify things: we could write any equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ as

$$\vec{a} \cdot \vec{x} = b,$$

where

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Suppose we have two equations from a system in this notation:

$$\vec{a} \cdot \vec{x} = b \tag{4.15}$$

$$\vec{c} \cdot \vec{x} = d. \tag{4.16}$$

Suppose $\vec{x}$ is a solution to both of these equations. Then for any nonzero scalar α,

$$\begin{aligned} \vec{c} \cdot \vec{x} + \alpha b &= d + \alpha b && \text{since } \vec{c} \cdot \vec{x} = d, \\ \vec{c} \cdot \vec{x} + \alpha(\vec{a} \cdot \vec{x}) &= d + \alpha b && \text{since } \vec{a} \cdot \vec{x} = b, \text{ and} \\ (\vec{c} + \alpha\vec{a}) \cdot \vec{x} &= d + \alpha b && \text{by distributive properties of inner product.} \end{aligned}$$

Thus, $\vec{x}$ is also a solution to the sum of equations (4.15) and (4.16). Now suppose $\vec{x}$ is a solution to the system

$$\vec{a} \cdot \vec{x} = b \tag{4.17}$$

$$(\vec{c} + \alpha\vec{a}) \cdot \vec{x} = \alpha b + d. \tag{4.18}$$

Then for any nonzero scalar α,

$$\begin{aligned} (\vec{c} + \alpha\vec{a}) \cdot \vec{x} - \alpha b &= d && \text{by Equation (4.18)}, \\ (\vec{c} + \alpha\vec{a}) \cdot \vec{x} - \alpha(\vec{a} \cdot \vec{x}) &= d && \text{since } \vec{a} \cdot \vec{x} = b, \text{ and} \\ \vec{c} \cdot \vec{x} &= d && \text{by distributive properties of inner product.} \end{aligned}$$

Thus, $\vec{x}$ is also a solution to $\vec{c} \cdot \vec{x} = d$. Since $\vec{x}$ is a solution to $(\vec{c} + \alpha\vec{a}) \cdot \vec{x} = d + \alpha b$ if and only if it is a solution to both $\vec{a} \cdot \vec{x} = b$ and $\vec{c} \cdot \vec{x} = d$, we have

6: These operations should remind you of something if you have not forgotten everything we talked about in Section 1.1.

Or if you haven't skipped ahead to Chapter 4 because matrices are your favorite topic…

What!? That's not allowed!

Actually, it is…There's a note in the introduction and everything.

7: You may note that Gauss-Jordan elimination fans on the internet often neglect proving this all-important fact.

No!

It's true.

that the following systems have the same set of solutions:

$$\begin{array}{rcl} \vec{a}\cdot\vec{x} & = & b \\ \vec{c}\cdot\vec{x} & = & d \end{array} \qquad \begin{array}{rcl} \vec{a}\cdot\vec{x} & = & b \\ (\vec{c}+\alpha\vec{a})\cdot\vec{x} & = & d+\alpha b \end{array}$$

□

Corollary 4.1.3 *One may apply any number of the operations in Theorem 4.1.2 in any order to a system of equations without changing the set of solutions for that system.*

Since a system of equations can also be represented as an augmented matrix, we can do the analogous operations to a matrix.

Definition 4.1.2 *Let $A \in \mathcal{M}_{m\times n}$. The following manipulations of A are called* ***row operations****:*

(a) *interchanging any two rows in A;*
(b) *multiplying any row by a nonzero scalar; and*
(c) *replacing the ith row with the sum of the ith row and any nonzero scalar multiple of any of the other rows.*

Any matrix resulting from any row operation on A is called ***row equivalent*** *to A.*[8]

Corollary 4.1.4 *Let $[A|\vec{b}]$ be the augmented matrix for a given system of m equations in n variables. Row operations on $[A|\vec{b}]$ do not change the set of solutions for the associated system.*

PROOF. This follows immediately from Theorem 4.1.2 and Theorem 4.1.1. □

Corollary 4.1.4 and augmented matrices are the primary means by which systems of equations are solved in most linear algebra courses, so we should probably work on that a bit. Keep in mind, though, that Theorem 4.1.1 provides multiple ways to represent a system of equations, so there may be situations where other methods are preferred.

For organizational purposes, we need to pick a notational convention for row operations. Here's one:

(a) $A \xrightarrow{\vec{r}_i \leftrightarrow \vec{r}_j} B$ means interchanging row i and row j in A results in the matrix B;

(b) $A \xrightarrow{a\vec{r}_i \to \vec{r}_i} B$ means multiplying row i by $a \neq 0$ results in the matrix B; and

(c) $A \xrightarrow{a\vec{r}_i+\vec{r}_j \to \vec{r}_j} B$ means replacing the jth row with the sum of the jth row and $a \neq 0$ times the ith row results in the matrix B.

This gives us a way to keep track of our operations, but most of us omit these once we've mastered row reduction.[9]

8: *"Row equivalent" is really nothing like "equivalent." Maybe there should be another word.*

What about "row similar"?

Noooo! Don't say that. "Similar" will mean something better than "row equivalent" in the next chapter, and now you've confused everyone!

It'll be fine. You're assuming they're reading these remarks. I'm sure no one saw that one... Also, who picks these words anyway?

9: You'll likely find yourself doing the steps of row reduction "in your head" sooner than you'd expect. At that point, though, you should use this notation to punish yourself when you make an error.

Example 4.1.1 Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_1 + 6x_2 + 7x_3 &= 8 \\ 10x_2 + 11x_3 &= 12. \end{aligned}$$

By Theorem 4.1.1, this system is equivalent to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{array}\right].$$

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{array}\right] &\xrightarrow{-5\vec{r}_1+\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 10 & 11 & 12 \end{array}\right] \\
&\xrightarrow{-1/4\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 10 & 11 & 12 \end{array}\right] \\
&\xrightarrow{-10\vec{r}_2+\vec{r}_3\to\vec{r}_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -9 & -18 \end{array}\right] \\
&\xrightarrow{-1/9\vec{r}_3\to\vec{r}_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]
\end{aligned}$$

We could be done here; we see from the last row of the last matrix that $x_3 = 2$, and we could substitute this into the other two equations to find x_1 and x_2 as well. Alternatively, we can just keep cooking with the row reduction.

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] &\xrightarrow{-2\vec{r}_3+\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \\
&\xrightarrow{-3\vec{r}_3+\vec{r}_1\to\vec{r}_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \\
&\xrightarrow{-2\vec{r}_2+\vec{r}_1\to\vec{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]
\end{aligned}$$

Well, *that's* easy to read. We have $x_1 = 0$, $x_2 = -1$, and $x_3 = 2$, which is the unique solution to the original system.

Target Format: Reduced Row-Echelon Form

Note that Corollary 4.1.3 implies that one has a lot of options when applying row operations to an augmented matrix. This can be a bit overwhelming. It is entirely possible to apply one thousand row operations (correctly) to an augmented matrix only to get the augmented matrix you started with. This can be a bit frustrating. We should avoid feeling overwhelmed and frustrated. It will help us to have a target in mind for our row operations.

Definition 4.1.3 *Let $A \in \mathcal{M}_{m\times n}$. We say the matrix A is in* ***row-echelon form*** *if*

- *the first nonzero number from the left, also called the* ***pivot****, of any nonzero row is always strictly to the right of the pivot of the row above, and*
- *any row with nonzero entries is above any row of all zeros.*

We say A is in ***reduced row-echelon form*** *if*

- *it is in row-echelon form,*
- *every pivot is a 1, and*
- *every pivot is the only nonzero entry in its column.*

This probably seems like a really weird definition; it definitely reads more like a tax form than most definitions. Nevertheless, it will have its uses. Let us explore the echelon-iness of some matrices.[10]

Example 4.1.2 Here are some matrices. Let's determine whether they are in row-echelon form, reduced row-echelon form, or neither.

- $\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

  This matrix is in neither form. In fact, it does not appear to be row reduced whatsoever.
- $\begin{bmatrix} 4 & 2 & 4 & 8 \\ 0 & 10 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  These matrices are all in row-echelon form. The pivots have been colored red.[11]
- $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  These matrices are all in reduced row-echelon form. Almost all the pivots have been colored red; one rogue pivot[12] has been colored blue for no good reason at all.

Exploration 105 The matrices in each bullet list below are row reductions of the same matrix. Circle the row-echelon form matrices. Put a square around the matrix in reduced row-echelon form. Highlight your favorite column vector with a pink sparkle pen.

- $\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix},$

10: What does "echelon" mean?
Model name of a Ford sedan from the 1970's?
No.
I think there's a sci-fi series-
No.
Michael Jordan's middle name?
No! I'm quite sure it's not a proper noun!

11: Do I have to pronounce the t in "pivot"?
Yes. Can I pronounce it with a long i?
Sure. What could go wrong?

12: Rogue Pivot! Dibs on band name!
Only with a long i and a silent t.
I hate to admit it, but that is a solid name.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}$$

► $$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, consider if these had been augmented matrices with the bar before the final column missing. Which form is the easiest to use when determining the solution of your system of linear equations?

Indeed, reduced row-echelon form is the form in which it is arguably[13] easiest to find the solution(s) to the associated system, and there are other benefits we shall uncover later.

13: I will fight you.

Now that we've talked about reduced row-echelon form and implied that this is the target format from the subsection title, you might be wondering whether this target is actually always achievable. Well, it is. Here's a theorem that says so, so it must be true.[14]

14: This is a classic example of "proof by intimidation." The idea here is that if you argue something with strong enough language, even without sound logic, the reader might believe it.

Theorem 4.1.5 *Suppose $A \in \mathcal{M}_{m\times n}$. Then there exists a unique matrix $B \in \mathcal{M}_{m\times n}$ in reduced row-echelon form that can be obtained from A by performing row operations.*

The proof of this statement relies on the technique of mathematical induction and Gauss-Jordan Elimination, so we will leave it to the Appendix.[15] Note here that it is *reduced* row-echelon form that is unique. In fact, scaling any row in a row-echelon form matrix produces a different row-echelon matrix associated to the same system of equations. Perhaps you are wondering why we even bother defining row-echelon form then? Well, the columns which contain pivots can be identified once the matrix is placed in row-echelon form, and there are some situations where the only information needed to answer a question is this. We'll see examples of these types of questions in the next two sections in particular. For now, though, it really will be *reduced* row-echelon form that is our target.

15: Oh, good. I was worried we were just going to declare it as fact without proof.

Procedure for Gauss-Jordan Elimination

We mentioned Gauss-Jordan elimination before, but never precisely defined it. That's because we needed a bit of terminology. **Gauss-Jordan elimination** is an algorithm that uses only the row operations to put any matrix into reduced row-echelon form. Thus, by Corollary 4.1.4, the resulting reduced row-echelon matrix has the same solution as the original matrix. It is definitely a thing you (or a computer!) could do. Here are the steps:

(1) Move any row of all zeros to the bottom of the matrix.

(2) Start at the leftmost column that has a nonzero entry in a row not already containing a reduced pivot. Designate an entry in this column to be a pivot and scale that row so that the pivot is a 1.

(3) Move the row of the pivot you are currently working with to be the highest row without a pivot already in reduced position.

(4) Use this pivot to produce zeros in every entry below it in the same column (by adding appropriate multiples of the pivot's row to the rows below).

(5) Repeat Steps 1–4.

(6) Continue this process until there are no more columns satisfying Step 2.

(7) Now begin at the bottom nonzero row and use the pivot to produce zeros (as in Step 3) in every entry *above* it in the same column.

(8) Move up one row and repeat Step 7.

(9) Repeat Step 8 until there are no more rows.

Example 4.1.3 Example 4.1.1 was this exact procedure. Go look at it again. Did you look? Seriously, you have to go look at it.

Example 4.1.4 Fine. We'll do another one. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

(a) Move any row of all zeros at the bottom of the matrix. Alright::

$$A \xrightarrow{\vec{r}_1 \leftrightarrow \vec{r}_4} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 4 & 6 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Start at the leftmost column that has a nonzero entry in a row not already containing a reduced pivot and designate a pivot. Scale this pivot's row so that the pivot is 1 (by multiplying the row by the appropriate nonzero scalar). Already done! Take that, Gauss! You're not the boss of me!

(c) Move the row of the pivot to be the highest row without a pivot already in reduced position. Hmmm... That seems to already be done as well. We are rocking this!

(d) Use this pivot to produce zeros in every entry below it in the same column (by adding appropriate multiples of the pivot's row to the rows below). So ordered:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 4 & 6 & -1 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2\vec{r}_1 + \vec{r}_2 \to \vec{r}_2} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & -3 & 8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Repeat steps 1–4. Ugh, that's a lot to ask.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 2 & 4 & -3 & 8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{1/2\vec{r}_2 \to \vec{r}_2} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3/2 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Well, I guess that wasn't so bad.

(f) Continue this process until there are no more columns satisfying Step 2. Hey, all those zeros and ones were pretty convenient!

(g) Now begin at the bottom nonzero row and use the pivot to produce zeros (as in step 4) in every entry *above* it in the same column.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3/2 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{3/2\vec{r}_3 + \vec{r}_2 \to \vec{r}_2} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\vec{r}_3 + \vec{r}_1 \to \vec{r}_1} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(h) Move up one row and repeat step 6.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\vec{r}_2 + \vec{r}_1 \to \vec{r}_1} \begin{bmatrix} 1 & 0 & -1 & 0 & -5 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Victory!

(i) Repeat step 8 until there are no more columns satisfying Step 2. Thanks for the advice, but we already declared victory.

Exploration 106 Let's play a bit of fill in the blanks. We'll row reduce a matrix using Gauss-Jordan Elimination, but leave out the names of the row operations. You fill in the operation that was done at each step. Ready? Go!

$$\begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & 3 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{\_\_\_\_} \to \text{\_\_\_\_} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Section Highlights

- A system of equations can be represented equivalently as a vector equation, a matrix equation, or an augmented matrix. See Theorem 4.1.1.
- There are three types of row operations that can be done to a matrix as part of Gauss-Jordan Elimination:
  - Rows can be scaled by any nonzero real number.
  - Rows can be swapped.
  - Any row can be replaced by a linear combination of other rows.

  See Definition 4.1.2.
- Gauss-Jordan Elimination is a process that uses row operations to transform a matrix into reduced row-echelon form. See Definition 4.1.3 and Example 4.1.4.
- If row operations from Gauss-Jordan Elimination are used on an augmented matrix, then the system of equations corresponding to the resulting matrix has the same set of solutions as the system corresponding to the original augmented matrix. This gives a convenient way to solve a system of equations since reduced row-echelon form makes the solution easier to identify. See Corollary 4.1.4 and Example 4.1.1.

Exercises for Section 4.1

4.1.1. Write the system as a vector equation, a matrix equation, and an augmented matrix.

(a) $$\begin{aligned} x_1 + 3x_2 &= 5 \\ 3x_1 &= 2 \\ x_1 - x_2 &= 1 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ 3x_1 + x_3 &= 2 \\ x_2 - x_3 &= 0 \end{aligned}$$

(c) $$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ 3x_1 + 2x_3 + x_5 &= 2 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

(d) $$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 1 \\ 3x_1 + x_3 + x_4 &= 1 \\ x_2 - x_3 &= 1 \end{aligned}$$

4.1.2. Determine whether the matrices below are in row-echelon form, reduced row-echelon form, or neither.

(a) $$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.3. Write down every possible reduced row-echelon matrix in $\mathcal{M}_{2\times 4}$. You may use any symbol you like to represent real numbers that are neither 1 nor 0.

4.1.4. Write down every possible reduced row-echelon matrix in $\mathcal{M}_{3\times 3}$. You may use any symbol you like to represent real numbers that are neither 1 nor 0.

4.1.5. Determine whether or not the pair of matrices is row equivalent. If they are list the row operations that transform the former into the latter.

(a) $$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

4.1.6. Row reduce the matrices to obtain reduced row-echelon form. (That is, perform Gauss-Jordan Elimination.)

(a) $\begin{bmatrix} 1 & -2 \\ 1 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -2 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -4 \\ 3 & 0 & -6 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 6 & 6 \\ 0 & 0 & 6 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 0 & 6 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 1 & -1 & 1 & 2 \\ 2 & 2 & -2 & 1 & 0 \\ 0 & 0 & 6 & 1 & 4 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 6 & 6 \\ 0 & 3 & 0 & 6 \end{bmatrix}$

(j) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 6 & 6 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

(k) $\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 6 & 2 & 6 \\ 0 & 3 & 0 & 6 & 0 \end{bmatrix}$

4.1.7. Is V from Equation 4.14, the set of all linear equations in n variables with real coefficients, a vector space? Justify your argument.

4.2 More Systems of Equations and Matrices

Now that we have a new method to solve a system of equations, we should spend some time talking about what a solution might look like and say a bit more about how matrices help us find them.

Parametric Solutions to Systems

Augmented matrices in reduced row-echelon form have the wonderful property that one can "read" the solution to the associated system right off the matrix. Just look back at Example 4.1.1. There is a small subtlety, though.

Example 4.2.1 Let's solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 5x_1 + 6x_2 + 7x_3 &= 8 \\ 9x_1 + 10x_2 + 11x_3 &= 12. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow[]{\substack{-5\vec{r}_1 + \vec{r}_2 \to \vec{r}_2 \\ -9\vec{r}_1 + \vec{r}_3 \to \vec{r}_3}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array}\right]$$

$$\xrightarrow{-2\vec{r}_2+\vec{r}_3\to\vec{r}_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-1/4)\vec{r}_2\to\vec{r}_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2\vec{r}_2+\vec{r}_1\to\vec{r}_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The final matrix above is in reduced row-echelon form. Now we can solve the system. The top row says $x_1 - x_3 = -2$ and the second row says $x_2 + 2x_3 = 3$. Solving for x_1 and x_2 gives us

$$x_1 = x_3 - 2$$

$$x_2 = -2x_3 + 3.$$

In terms of vectors, this would be all vectors of the form

$$\begin{bmatrix} x_3 - 2 \\ -2x_3 + 3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

where $x_3 \in \mathbb{R}$.

The augmented matrix in reduced row-echelon form in the previous example did not have a pivot in every variable column. That seems somewhat annoying, but it happens a lot. How did we deal with it in the example? We were able to solve for all the variables associated to a column containing a pivot, and some of those variables were in terms of the variable not associated to a pivot. This is not a coincidence.

Definition 4.2.1 *A **pivot variable** is a variable in a system of equations whose column in the associated augmented matrix in reduced echelon form contains a pivot. A **free variable** is a variable in a system of equations that is not a pivot variable. That is, a free variable in a system of equations is one whose column in the associated augmented matrix in reduced echelon form does not contain a pivot.*

Exploration 107 Solve for each of the pivot variables in terms of free variables in the augmented matrices below.

- $$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- $$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Definition 4.2.2 *A **parametric solution** for a system of m equations in n variables that has an infinite number of solutions is a representation of the solutions in which the pivot variables are given in terms of the free variables (often called **parameters**).*

Your answers from the exploration above were all parametric solutions to the systems represented by the matrices. We will often also make note of which variables are free variables in solutions of this form.

Example 4.2.2 Consider the augmented matrix below.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 5 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solving for the pivot variables in terms of the free variables and taking note of which variables are free gives us a solution that looks like

$$\begin{gathered} x_1 = 2 - x_4, \\ x_2 = -1 - x_4 - 5x_3, \\ x_3 \text{ free}, \\ x_4 \text{ free}. \end{gathered}.$$

We could write this in set notation with vectors as

$$\left\{ x_3 \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}.$$

Zero, One, or Many

The Gauss-Jordan elimination procedure and row-echelon form make determining whether a system has a solution pretty simple. First of all, when an augmented matrix is in even just row-echelon form, we can tell quickly that a solution does not exist.

Theorem 4.2.1 *A system of m equations in n variables has no solutions if the associated augmented matrix in row-echelon form has a pivot in the last column.*

PROOF. Let $[A|\vec{b}]$ be the associated augmented matrix in row-echelon form. Suppose $[A|\vec{b}]$ has a pivot in the last row. Then this row corresponds to an equation in the associated system of the form

$$0x_1 + 0x_2 + \cdots + 0x_n = b_i,$$

for some $b_i \neq 0$. This equation has no solutions, so any system of which it is a part also has no solution. □

Now, if we do not have a pivot in the final column, there are actually two possibilities.

Theorem 4.2.2 *A system of m equations in n variables has a unique solution if the associated augmented matrix in reduced row-echelon form has a pivot in every column except the last.*

PROOF. Suppose $[A|\vec{b}]$ is the augmented matrix in reduced row-echelon form. We note that any nonzero row of this matrix has a 1 in a single pivot location and a 0 in all remaining locations except the final one. Thus, any associated equation will be of the form

$$x_i = b_i$$

where x_i is a variable and b_i is the i^{th} entry in the final column $\vec{b}$. Thus, each variable is determined by the entries of $\vec{b}$, and there is a unique solution. □

Theorem 4.2.3 *A system of m equations in n variables has infinitely many solutions if the associated augmented matrix in reduced row-echelon form has no pivot in the last column and at least one other column. That is, the system has infinitely many solutions if it has a free variable and no pivot in the final column.*

PROOF. Suppose $[A|\vec{b}]$ is the augmented matrix in reduced row-echelon form. Any nonzero row of this matrix has a 1 in a single pivot location and a 0 in all remaining pivot locations. Thus, we can solve for the pivot variable in terms of the free variables and the entry in our solution column. That is, if x is a pivot variable, there is some row of $[A|\vec{b}]$ whose associated equation is

$$x + a_1y_1 + \cdots + a_ky_k = b$$

where $a_1, \ldots, a_k \in \mathbb{R}$, $y_1, \ldots y_k$ are free variables, and b is the corresponding entry in the solution column. This allows us to say

$$x = b - a_1 y_1 - \cdots - a_k y_k.$$

Since we assumed there is no pivot in the final column, each row containing a pivot can be treated thusly, and doing so gives us a full parametric solution for the system of equations. All rows not containing a pivot are rows of zero since we are in reduced row-echelon form. Then, we can choose a real number for each free variable. Any choice gives a valid solution, and there are infinitely many possible choices for these real numbers. Furthermore, each distinct choice gives rise to a distinct solution since the free variables themselves would then be distinct. Thus, there are infinitely many solutions. □

Combining these three theorems, we can state the following:

Corollary 4.2.4 *A system of m equations in n variables has either zero solutions, one solution, or infinitely many solutions.*

Corollary 4.2.5 *A system of m equations in n variables has no solutions if and only if the associated augmented matrix in row-echelon form has a pivot in the last column.*

Note also that Theorem 4.2.2 and Theorem 4.2.3 stated that the matrix was in *reduced* row-echelon form. This is really just to make the proofs clearer. Since the locations of pivots can be identified once the matrix is in row-echelon form, we could have stated these theorems using the weaker condition instead. Therefore, if you are trying to answer the question of how many solutions rather than actually finding them, you could actually save yourself some time by only reducing the matrix to row-echelon form.

Exploration 108 The matrices below are the examples of row-echelon form from the previous section. Cross out the ones that correspond to systems with no solution. Circle the ones with a unique solution. Do the ones left have infinitely many solutions?

$$\left[\begin{array}{ccc|c} 4 & 2 & 4 & 8 \\ 0 & 10 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \text{ and } \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Exploration 109 The augmented matrices below are in row-echelon form. Determine whether the corresponding systems of equations have no solution, a single solution, or infinitely many solutions. If there are infinitely many, write the parametric solution.

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$

Geometry of Solutions

There are a lot of subtle misconceptions about the relationship between linear equations and their graphical representations. For example, it's very common to hear the equation $y = mx + b$ referred to as "a line." It's not that this is completely *wrong*, but it's definitely inaccurate. An equation and a line are two different things. Before we proceed, we should establish some concrete lingo.

Definition 4.2.3 *A* ***solution*** *for a linear equation $a_1x_1 + \cdots + a_nx_n = b$ is an n-tuple $(x_1, \ldots, x_n)$ that makes the linear equation true. The* ***graph*** *of a linear equation $a_1x_1 + \cdots + a_nx_n = b$ is a visual representation of the set of all n-tuples $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that make the linear equation true.*

There is a not-so-subtle distinction between the linear equation $a_1x_1 + \cdots + a_nx_n = b$, its set of solutions, and its graph. These three things are respectively an algebraic equation (that may or may not be true), a set of points, and a visual representation of that set of points.

Let us explore the consequences of this definition for systems. A solution for a system must be a solution for each equation in the system. We now have multiple ways of thinking about this:

- The solution for a system is the intersection of the sets of solutions for each equation in the system.
- The graph of the solution for a system is a visual representation of this intersection. That is, the graph of the solution for a system is where the graphs of each equation in the system intersect.

We need at least two variables for this to be the slightest bit interesting, so we'll start there.

Two Variable Cases:

- Two variables, one equation:
  Classic. If we have $a_1x_1 + a_2x_2 = b$, we could write
  $$x_2 = -\frac{a_1}{a_2}x_1 + \frac{b}{a_2},$$
  the standard "slope-intercept" form. Note that since we're defining x_2 as a function of x_1, we need $a_2 \neq 0$ to have an x_2, which works out really well since we also have to divide by a_2. In this case, there's always an infinite number of solutions in $\mathbb{R}^2$ for one linear equation in two variables. The graph of this equation, the set of all solutions to the equation, forms a line in the $\mathbb{R}^2$ plane.

- Two variables, two equations:
  This is significantly more interesting. With two equations, we have an infinite number of solutions for each equation. However, since a solution for a system must be a solution for *all* equations in the system, there are three possibilities. The set of solutions for each equation is a line in the plane, and these lines may be parallel, overlap completely, or intersect at one point. These three cases correspond to the system having infinite solutions, no solutions, or exactly one solution, respectively. See Figure 4.1.

- Two variables, three or more equations:
  This is not all that different from the previous case; the graph of the third equation simply provides a third line in the $\mathbb{R}^2$ plane. Any additional equations do the same. Convince yourself that we still only have three options for this system: infinite solutions, no solutions, or exactly one solution.

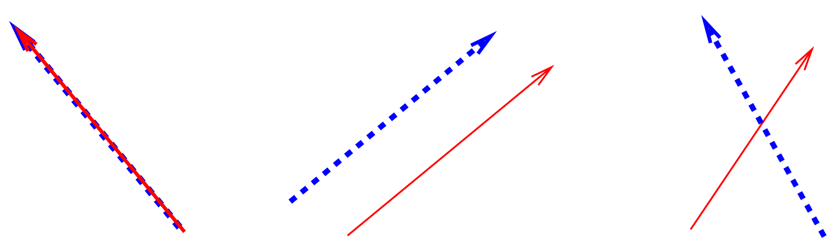

Figure 4.1: There are only three possible ways for two lines in $\mathbb{R}^2$ to intersect.

Three Variable Cases:

- Three variables, one equation:
  This is an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = b$, and the graph of this equation is a plane in $\mathbb{R}^3$. We can easily find y_1, y_2, y_3 such that $b = -(a_1y_1 + a_2y_2 + a_3y_3)$. Then $a_1x_1 + a_2x_2 + a_3x_3 = b$ if and only if
  $$a_1(x_1 - y_1) + a_2(x_2 - y_2) + a_3(x_3 - y_3) = 0,$$
  or $\vec{a} \cdot (\vec{x} - \vec{y}) = 0$, where
  $$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Since $\vec{a} \cdot (\vec{x} - \vec{y}) = 0$ if and only if $\vec{a}$ and $\vec{x} - \vec{y}$ are orthogonal, we have that $(x_1, x_2, x_3) \in \mathbb{R}^3$ is a solution for $a_1x_1 + a_2x_2 + a_3x_3 = b$ if and only if $\vec{x} - \vec{y}$ is orthogonal to $\vec{a}$. Thus, the graph of $a_1x_1 + a_2x_2 + a_3x_3 = b$ is the plane of points orthogonal to $\vec{a}$, containing the point (y_1, y_2, y_3).

- Three variables, two equations:
  Now we have two planes, each the graph of one of the equations, in $\mathbb{R}^3$. How can two planes intersect in $\mathbb{R}^3$? They can be the same plane, parallel planes, or intersect in a line. Thus, we have infinite solutions for the system (a plane of solutions or a line of solutions) or no solutions.

- Three variables, three equations:
  This one's nice. There are a lot of ways three planes can intersect in $\mathbb{R}^3$. See Figure 4.2 for a couple of them. The three planes could coincide, yielding a plane of solutions for the system. All three planes could intersect in a line, and all three planes could intersect in a point. Lastly, if any two of the planes are parallel, then there are no solutions for the system.

- Three variables, four or more equations:
  Well, now there are planes shooting all over the place. Good luck drawing this one. It'll probably be harder to make them all intersect, but it's not hard to arrange cases where we have a plane of solutions for the system, a line of solutions for the system, a single point for the solution, or no solution for the system.

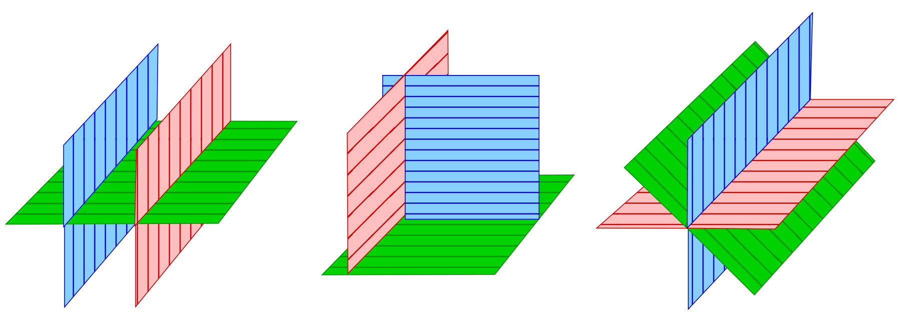

Figure 4.2: There are many ways for three planes in $\mathbb{R}^3$ to intersect.

The Many Dimensions of "Infinitely Many"

We saw before that when a system has a solution and a free variable, there are infinitely many solutions. Yet, in the examples we just looked at, we see that perhaps there is an infinite "line-worth" of solutions, or perhaps there is an infinite"plane-worth" of solutions. Actually, there's a bit more we can say in the case where there are infinitely many solutions.

Theorem 4.2.6 *If a system of m linear equations in n variables has a solution, then the set of solutions is in one-to-one correspondence onto a $k-$dimensional subspace of $\mathbb{R}^n$, where k is the number of free variables in the reduced row-echelon form of the coefficient matrix associated to the system.*

Now, in the case that there are no free variables, this statement reduces naturally to the statement that there is a unique solution. What's really interesting here is that we get a concept of size based on the number of free variable in the situation of infinitely many solutions. The proof of this statement gets a bit off track, so we have moved it to the Appendix. However, we can show you an example that will help illustrate the idea of the proof.

Example 4.2.3 Let's start with a system of equations.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 3 \\ 2x_1 + 2x_2 + x_3 + 4x_4 &= 3 \\ 4x_1 + 4x_2 + 2x_3 + 8x_4 &= 6 \end{aligned}$$

Great! Now, let's make that into an augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 4 & 3 \\ 4 & 4 & 2 & 8 & 6 \end{array}\right]$$

Wonderful! Now let's row reduce to reduced row-echelon form. We'll skip the steps, just to keep this example short.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Stupendous! We can see from this reduced row-echelon form that x_2 and x_4 are free variables and x_1 and x_3 are pivot variables. Moreover, we can write out a parametric solution for this system.

$$\begin{aligned} x_1 &= 1 - x_2 - 2x_4 \\ x_3 &= 1 \\ x_2 &\text{ free} \\ x_4 &\text{ free} \end{aligned}$$

This says the set of solutions to this system is

$$U_{\text{soln}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \colon x_2, x_4 \in \mathbb{R} \right\}.$$

By Theorem 4.2.6, we expect then that there is a 2-dimensional subspace in one-to-one correspondence with this set U_{soln}. The 2-dimensional subspace is actually the kernel of a matrix. Define

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the matrix obtained by just omitting the last column of the reduced row-echelon augmented matrix. To find $\text{Ker } A$, we need to solve $A\vec{x} = \vec{0}$. Since A is in reduced row-echelon form and the augmented column will be all 0's, we can go straight to a parametric solution.

$$\begin{aligned} x_1 &= -x_2 - 2x_4 \\ x_3 &= 0 \\ x_2 &\text{ free} \\ x_4 &\text{ free} \end{aligned}$$

That is,

$$\text{Ker } A = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \colon x_2, x_4 \in \mathbb{R} \right\}.$$

By comparing U_{soln} to Ker A we see that U_{soln} is actually a *shifted* version of Ker A. The one-to-one and onto correspondence is then just the function that shifts Ker A appropriately; specifically, the map $S\colon \text{Ker } A \to U_{\text{soln}}$ given by

$$S(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is the invertible shift function.[16] It follows that geometrically, the solution space, U_{soln}, is a plane that does not go through the origin in $\mathbb{R}^4$.

16: Is there a joke to be made about S being "shifty"?

Ugh. Let's just assume you made it. Well done. Moving on…

Did you two catch that S is not an isomorphism since U_{soln} is not a subspace of $\mathbb{R}^4$?

Exploration 110 Describe the solution space for the system geometrically if possible.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 3 \\ 2x_1 + 2x_2 + 4x_4 &= 3 \\ 3x_1 + 3x_2 + 2x_3 + 4x_4 &= 0 \end{aligned}$$

Let's think for a minute about what Theorem 4.2.6 tells us about the kernel of a matrix. Suppose $A \in \mathcal{M}_{m\times n}$. Then finding Ker A is equivalent to solving the matrix equation $A\vec{x} = \vec{0}$. Thus, there is a system of m equations and n unknowns involved here, and Ker A is the solution space for this system.

Corollary 4.2.7 *If $A \in \mathcal{M}_{m\times n}$ row reduces to a matrix with k free variables, then* $\dim \text{Ker } A = k$.

Additionally, we know that if A is a matrix representation for some linear transformation $T\colon V \to W$, then Ker T is isomorphic to Ker A. We also know $\dim \text{Imag } T$ can be determined once $\dim \text{Ker } T$ is known because of the Rank-Nullity Theorem.

Corollary 4.2.8 *Suppose $T\colon V \to W$ is a linear transformation with matrix representation $A \in \mathcal{M}_{m\times n}$. Then $\dim \operatorname{Ker} T$ is the number of free variables identified once A is row reduced and $\dim \operatorname{Imag} T$ is the number of pivot variables.*

Section Highlights

- A system of equations has either one solution, infinitely many solutions, or no solutions. See Corollary 4.2.4.
- The number of solutions for a system of equations can be quickly determined by examining the pivots in the row-echelon form of the corresponding augmented matrix. In particular:
  - If there is a pivot in the augmented column, then the system has no solutions.
  - If there is a pivot in all columns except the final augmented column, then the system has exactly one solution.
  - If neither of the above two situations occur, then the system has infinitely many solutions.

  See Corollary 4.2.5, Theorem 4.2.2, and Theorem 4.2.3.
- If a system has infinitely many solutions, then they can be described parametrically in terms of the free variables. See Definition 4.2.2 and Example 4.2.2.

Exercises for Section 4.2

4.2.1. Determine by inspection whether the system represented by the augmented matrix has no solution, one solution, or infinitely many solutions.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

4.2.2. For each augmented matrix in reduced row-echelon form below, give either the unique solution or the parametric solution.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

(e) $\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 1 & 4 \\ 0 & 1 & 5 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$

(g) $\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & 5 & 2 \\ 0 & 1 & 0 & 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

4.2.3. Determine the value (or values) of k for which the system will have zero, exactly one, or infinitely many solutions, if possible.

(a) $\begin{array}{rcrcr} 2x & + & ky & = & 6 \\ 3x & + & 6y & = & 0 \end{array}$

(b) $\begin{array}{rcrcr} 2x & + & ky & = & 0 \\ x & + & 4y & = & 0 \end{array}$

(c)
$$\begin{array}{rcrcrcl} x & - & ky & + & z & = & k \\ x & + & 2y & - & 2z & = & k \\ 2kx & + & 2y & - & 2z & = & 2k. \end{array}$$

4.2.4. Show that for any $k_1, k_2 \in \mathbb{R}$, if $ad - bc \neq 0$, then

$$\begin{array}{rcrcl} ax & + & by & = & k_1 \\ cx & + & dy & = & k_2 \end{array}$$

has a unique solution and state the solution.

4.2.5. Find the line of intersection for the planes given by the equations in the system.

(a)
$$\begin{aligned} -2x - 7y + z &= 12 \\ x + 2y + 4z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} x - y + z &= 3 \\ 3x - 3y &= 6. \end{aligned}$$

4.2.6. Set up an augmented matrix and row reduce to solve the system of equations.

(a)
$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 + 2x_2 &= 2 \\ x_1 + x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 + 2x_2 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 - x_2 &= -4 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + x_3 &= 4 \\ -x_1 + 2x_2 &= 2 \\ 3x_2 + x_3 &= 3 \end{aligned}$$

(e)
$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ -x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + 5x_2 + 5x_3 &= 5 \\ -x_1 + 2x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}$$

(g)
$$\begin{aligned} x_1 + 5x_2 + 5x_3 + x_4 &= 5 \\ -x_1 + 2x_2 + 3x_3 + x_4 &= 2 \\ 2x_1 + 3x_2 + 2x_3 &= 3 \end{aligned}$$

(h)
$$\begin{aligned} x_1 + 5x_2 + 6x_3 &= 5 \\ -x_1 + 2x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

(i)
$$\begin{aligned} x_1 + 5x_2 + 4x_3 &= 1 \\ -x_1 + 2x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

(j)
$$\begin{aligned} x_1 - 2x_2 + 5x_3 + x_4 + x_5 &= 2 \\ -x_1 + 2x_2 - x_3 + x_4 &= 2 \\ x_1 + 3x_2 + 2x_3 + x_5 &= 3 \end{aligned}$$

(k)
$$\begin{aligned} x_1 + 5x_3 + x_4 + x_5 &= 2 \\ -x_1 + 2x_2 - x_4 &= 2 \\ x_1 + 3x_2 + 2x_3 + x_5 &= 3 \\ x_1 + 2x_4 + x_5 &= 0 \end{aligned}$$

4.3 Matrix Techniques

Gauss-Jordan elimination was a useful tool for finding solutions for a system of equations in the last section. Now, think back to all those times we've needed to solve a system of equations so far. All the work to learn this technique is about to pay off in a huge way.

It should also be noted that technology is often quite useful and helpful when implementing Gauss-Jordan elimination.[17] As with all technology, there are advantages (efficiency!) and disadvantages (potential for inaccuracy!), but it is not difficult to find reasonably simple-to-use matrix calculators.[18]

17: You mean with robots?
No, not specifically robots.

18: Robots!
No, Ricky! Not robots!

Linear Independence

If we'd like to determine whether or not a set is linearly independent, there is a matrix we can construct and row reduce to answer this question.

Theorem 4.3.1 *A set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{R}^n$ is linearly independent if and only if the matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$ row reduces to a matrix with a pivot in every column.*

PROOF. We know that the set S is linearly independent if and only if the only solution to

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0} \tag{4.19}$$

is the trivial solution with $x_1 = x_2 = \cdots = x_n = 0$. Let $A = [\vec{v}_1 \cdots \vec{v}_n]$. We can then rewrite Equation 4.19 as $A\vec{x} = \vec{0}$ where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

We know this equation has the trivial solution, so whether or not that is the only solution comes down to whether there are free variables. Now, we can row reduce A. By Theorem 4.2.2, we know this will be the only solution if and only if A has a pivot in every column. □

Example 4.3.1 Let us discover if the following set of vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Using these vectors as columns in a matrix, we have

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix},$$

which in reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 4.3.1, these vectors are not linearly independent since the last column does not contain a pivot.

Exploration 111 Recall a set of three vectors in $\mathbb{R}^3$ will be a basis if it is linearly independent since $\dim \mathbb{R}^3 = 3$. Use Theorem 4.3.1 to determine whether this set is a basis for $\mathbb{R}^3$. Note that row-echelon form is enough to identify the locations of pivots.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finding a Basis

In that last exploration, we used row reduction to identify a basis since we already knew the dimension of our space. What about when we don't know the dimension? Well, row reduction can do this for us too. However, we first need to define some terminology to help us out.

Definition 4.3.1 *The ith column in a matrix A is called* ***pivot column*** *if the ith column of the row-echelon form of A contains a pivot. Since the rows and columns of a matrix are each vectors, we will often refer to a pivot column as a vector.*

Note that a pivot column is the original column in the matrix *before* you put the matrix in row-echelon form. Doing Gauss-Jordan elimination will identify, *not yield*, which columns are pivot columns. For example, if you row reduce a matrix A and find that the resulting matrix B has a pivot in the third column. The third column of B is *not* a pivot column of A. However, since B has a pivot in the third column, we know that the third column of A is a pivot column.

Pivot columns, it turns out, are very important.[19]

Theorem 4.3.2 *Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set vectors in $\mathbb{R}^m$. The pivot columns of the matrix $[\vec{v}_1 \cdots \vec{v}_n]$ form a basis for* $\operatorname{Span}\{S\}$.

19: Yes, but what about pea-voh columns?

I think you're supposed to accent the second syllable in pea-*voh*.

PROOF. Let $A = [\vec{v}_1 \cdots \vec{v}_n]$. We can row reduce A to reduced row-echelon form and identify the pivot columns. For reference, we will denote this reduced row echelon form of A by B. From Theorem 4.3.1, we see that in the case where all the columns of A are pivot columns, the set S is linearly independent and forms a basis for $\mathrm{Span}\,\{S\}$. Suppose now that there are free variables, so that $A\vec{x} = \vec{0}$ or equivalently

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}. \tag{4.20}$$

has infinitely many solutions. Let us re-label the variables in Equation 4.20 so that we sort between pivot and free variables. We will use $y_1, \ldots, y_k$ to denote the pivot variables and $z_1, \ldots, z_l$ to denote the free variables. Then Equation 4.20 can be rearranged as

$$y_1\vec{v}_{y_1} + \cdots + y_k\vec{v}_{y_k} + z_1\vec{v}_{z_1} + \cdots + z_l\vec{v}_{z_l} = \vec{0}. \tag{4.21}$$

Recall that our matrix B is the reduced row-echelon form of A. Thus, each row is either a row of all zeros or it contains a pivot. Since we are trying to determine linear independence, we can augment our matrix A by a column of zeros, and the result for B will remain a column of zeros. We can then use our technique of parametric solutions to write each pivot variable as a linear combination of the free variables. That is, we can write

$$\begin{aligned} y_1 &= c_{11}z_1 + \cdots + c_{1l}z_l \\ &\vdots \\ y_k &= c_{k1}z_1 + \cdots + c_{kl}z_l \end{aligned}$$

for some real numbers c_{ij} determined by the entries of B. Now, any choice for the free variables gives one specific solution in our solution space for $A\vec{x} = \vec{0}$. If we choose $z_1 = -1$ and $z_2 = \cdots = z_l = 0$, then $y_1, \ldots, y_k$ are determined and give a solution to

$$y_1\vec{v}_{y_1} + \cdots + y_k\vec{v}_{y_k} - \vec{v}_{z_1} = \vec{0}.$$

This rearranges to give us

$$y_1\vec{v}_{y_1} + \cdots + y_k\vec{v}_{y_k} = \vec{v}_{z_1}.$$

So the free variable vector $\vec{v}_{z_1}$ can be written as a linear combination of the pivot column vectors $\vec{v}_{y_1}, \cdots, \vec{v}_{y_k}$. The same can be done with a similar choice for the free variables to solve for each of the vectors $\vec{v}_{z_i}$. Thus, the pivot column vectors $\vec{v}_{y_1}, \cdots, \vec{v}_{y_k}$ are a spanning set for $\mathrm{Span}\,\{S\}$.

To be a basis, we must also show that they are linearly independent. The easiest way to confirm this is to form the new matrix $C = [\vec{y}_1 \cdots \vec{y}_k]$ and perform the same row operations on C as those used to reduce A to B. Because of the definition of pivot columns, we will have a matrix with a pivot in every column, so the vectors $\{\vec{y}_1, \ldots, \vec{y}_k\}$ are linearly independent by Theorem 4.3.1. Therefore, the pivot columns are a linearly independent spanning set, a.k.a. a basis for $\mathrm{Span}\,\{S\}$. □

Example 4.3.2 Let's use this theorem to find a basis for a subspace. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Now, we can use a matrix and row reduction to find a basis for $\text{Span}\,\{S\}$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 0 \\ 4 & 0 & 4 & 1 & 5 & 0 \\ 3 & -1 & 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Wow, we row reduced that one fast! Okay, fine. We just omitted all of the steps. If you're looking for some additional row reduction practice, this would be a good one to try since you already have the answer. Now, let's talk about our basis for $\text{Span}\,\{S\}$. We see that columns 1, 2, 4, and 6 are the pivot columns, so

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\text{Span}\,\{S\}$.

Exploration 112 Use the technique from the previous example to find a basis for

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

One place we have needed to find a basis for a subspace in the past is when computing the column space of a matrix. The concept of pivot columns helps here naturally.

Example 4.3.3 Consider the matrix A defined below.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 3 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 0 & 1 \end{bmatrix}$$

Then we can row reduce to get

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -4 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Based on pivot columns, we see then that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

forms a basis for $\text{Col } A$. Recall that $\text{Col } A = \text{Imag } T_A$, where T_A is the linear transformation induced by the matrix A. Thus, this gives us a way to find the image of a linear transformation using matrix representations as well.

We've seen now that row reducing a matrix can help us find a basis for the span of a set of vectors. However, what would that look like when the set of vectors is a spanning set for the vector space? Well, to be a spanning set, the expected number of basis vectors identified would be at least the dimension of the vector space. Since the dimension of the vector space is the same as the number of rows in the matrix, we would see a pivot in every row of the corresponding matrix, since the number of pivot columns must match the dimension of the vector space.

Corollary 4.3.3 *Suppose* $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$ *is a subset of* $\mathbb{R}^n$. *If the matrix* $A = [\vec{v}_1 \ \ldots \ \vec{v}_m]$ *has a pivot in every row when row reduced, then* S *is a spanning set for* $\mathbb{R}^n$.

Linear Combinations

We can actually do better than just determining whether vectors are linearly dependent or not. If they are in fact linearly dependent, this matrix method provides a nice way to write one of the vectors as a linear combination of the others. To see this, let's look at an example.

Example 4.3.4 Let's just use the matrix from Example 4.3.2 for this one.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 0 \\ 4 & 0 & 4 & 1 & 5 & 0 \\ 3 & -1 & 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In our proof for Theorem 4.3.2, we saw that there was a way to solve for any free variable vector in terms of the pivot columns by choosing -1 for one free variable and 0 for the rest. Let's see what happens when we do this here. Using the convention of y's for pivot variables and z's for free variables as in the proof, our parametric solution for $A\vec{x} = \vec{0}$ becomes

$$y_1 = -z_1 - z_2 \tag{4.22}$$
$$y_2 = -z_1 \tag{4.23}$$
$$y_3 = -z_2 \tag{4.24}$$
$$y_4 = 0 \tag{4.25}$$

where y_1, y_2, y_3, and y_4 correspond to the pivot columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_4$, and $\vec{v}_6$ and z_1 and z_2 correspond to the free variable columns $\vec{v}_3$ and $\vec{v}_5$. Choosing $z_1 = -1$ and $z_2 = 0$ gives us a way to write $\vec{v}_3$ as a linear combination of the pivot columns, and choosing $z_1 = 0$ and $z_2 = -1$ gives us a way to write $\vec{v}_5$ as a linear combination of the pivot columns. Specifically, we have

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$$
$$\vec{v}_5 = \vec{v}_1 + \vec{v}_4.$$

Because of how nice reduced row-echelon form is, we can actually see the outcomes of these directly from the reduced free variable column.

Exploration 113 Form the same matrix as from Example 4.3.3 and then use the entries in the reduced free variable columns to write the free variable column vectors as linear combinations of the pivot columns. Check your answer by computing the linear combinations with the vectors.

Exploration 114 The following set of vectors are clearly linearly dependent. Set up a matrix and row reduce to find a way to write one of the vectors as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Coordinate Vectors

Now that we are talking about how matrices allow us to write vectors as linear combinations of basis vectors, we should really say something about coordinate vectors, too.

Recall that if we have a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for a vector space V, then for any vector $\vec{v} \in V$, we can write $\vec{v}$ as a unique linear combination of the vectors in $\mathcal{B}$; that is, $\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$ for some scalars $a_1, \ldots, a_n$. These scalars are the coordinates for $\vec{v}$ relative to $\mathcal{B}$:

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

We now have a very convenient method for finding these coordinates. We want to solve the vector equation

$$a_1\vec{b}_1 + \cdots + a_n\vec{b}_n = \vec{v} \tag{4.26}$$

for the scalars $a_1, \ldots, a_n$. If $V = \mathbb{R}^n$, then this is equivalent to the matrix equation and augmented matrix,

$$\begin{bmatrix} \vec{b}_1 \cdots \vec{b}_n \end{bmatrix} [\vec{v}]_{\mathcal{B}} = \vec{v} \quad \text{and} \quad \begin{bmatrix} \vec{b}_1 \cdots \vec{b}_n \mid \vec{v} \end{bmatrix},$$

respectively. This is also equivalent to a system of equations, but at this point, we suspect solving by way of the augmented matrix is the preferred method.

Example 4.3.5 Let

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

From the row reduction in Example 4.3.1 and Theorem 4.3.2, we know these three vectors

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

form a basis for H. Let's find the coordinates for

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

relative to $\mathcal{B}$. That is, we want to find scalars a_1, a_2, a_3 where

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3, \text{ so that } [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

This vector equation can be written as an augmented matrix and its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this we see that $\vec{v} = -\vec{b}_1 - 3\vec{b}_2 - 3\vec{b}_3$, so $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -3 \\ -3 \end{bmatrix}$.

Exploration 115 Consider the basis $\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

Write the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of these basis vectors.

What about if we want to find multiple coordinate vectors with respect to the same basis? This was what we had to do when finding the matrix representations for a linear transformation with respect to a non-standard basis. Well, we can augment with multiple vectors at once!

Example 4.3.6 Again consider the basis

$$\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ of } \mathbb{R}^3.$$

Suppose we have a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \\ T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \text{ and} \\ T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

In order to find the matrix representation for T with respect to the basis $\mathcal{B}_0$, we need now to convert the outputs given here into coordinate vectors for $\mathcal{B}_0$. Instead of row-reducing 3 separate times, we can do this all at once.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 \end{array}\right]$$

Then

$$A = \begin{bmatrix} -4 & -1 & 0 \\ 3 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

is the matrix representation for T with respect to the basis $\mathcal{B}_0$.

Lastly, can row reduction help us when our vector space is not $\mathbb{R}^n$ for some positive integer n? Of course it can. The key is finding coordinate vectors, and it's best to choose the standard basis since it's easier to work with. From there, we could apply any technique we want from the $\mathbb{R}^n$ setting and then translate our results back.

Example 4.3.7 Let's determine the coordinates for $\vec{p} = 2x^3 - 4x + 5$ relative to the vectors in the basis

$$\mathcal{B} = \left\{1, x, x^2 - 1, x^3 - 3x\right\}$$

for $\mathbb{P}_3$. Righto. We just need to find scalars $a_1, \ldots, a_4$ such that

$$a_1(1) + a_2(x) + a_3(x^2 - 1) + a_4(x^3 - 3x) = 2x^3 - 4x + 5.$$

Maybe that sounds like fun to you. If it does, great! There is, however, another way...

Recall that the standard basis for $\mathbb{P}_3$ is

$$\mathcal{E} = \left\{1, x, x^2, x^3\right\},$$

so we can write

$$[1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [x]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\left[x^2 - 1\right]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \left[x^3 - 3x\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly,

$$[\vec{p}]_{\mathcal{E}} = \left[2x^3 - 4x + 5\right]_{\mathcal{E}} = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 2 \end{bmatrix}.$$

Now we can turn the problem of finding scalars such that

$$a_1\,[1]_{\mathcal{E}} + a_2\,[x]_{\mathcal{E}} + a_3\left[x^2 - 1\right]_{\mathcal{E}} + a_4\left[x^3 - 3x\right]_{\mathcal{E}} = \left[2x^3 - 4x + 5\right]_{,}$$

or

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 2 \end{bmatrix}.$$

As we've seen, we can write this as the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right],$$

which, in reduced row-echelon form, is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right].$$

Thus, $a_1 = 5, a_2 = 2, a_3 = 0$, and $a_4 = 2$. Note that along the way we also verified that $\mathcal{B}$ is a basis for $\mathbb{P}_3$ since the columns in the reduced matrix corresponding to $\mathcal{B}$ all contain pivots and the set's size matches the dimension of $\mathbb{P}_3$.

Kernel of a Matrix

We've already discussed this in some examples in the previous section, but it's important enough to revisit. By definition,

$$\text{Ker } A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

for any $A \in \mathcal{M}_{m\times n}$. Now, solving $A\vec{x} = \vec{0}$ for $\vec{x}$ can be done using the augmented matrix representation of the matrix equation.

Example 4.3.8 Let's find $\text{Ker } A$ where

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}.$$

Then we row reduce this to get

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since we are solving $A\vec{x} = \vec{0}$, the column we would augment would only be a column of zeros, and such a column is never changed by row operations. Thus, we can omit the augmentation when solving $A\vec{x} = \vec{0}$. To find $\text{Ker } A$, we write the parametric solution.

$$\begin{aligned} x_1 &= x_4 \\ x_2 &= 3x_4 \\ x_3 &= 3x_4 \\ x_4 &\text{ free} \end{aligned}$$

Now, we can convert this to a set of solutions. In this case, we know the set of solutions will actually be a subspace since we know $\text{Ker } A$ is always a subspace. Thus,

$$\text{Ker } A = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} x_4 \colon x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Recall that for any linear transformation $T\colon V \to W$, we can find a matrix representation A. Then $\text{Ker } A$ can be translated by a coordinate mapping into $\text{Ker } T$, and thus, computing $\text{Ker } A$ can be an essential step for finding $\text{Ker } T$.

Example 4.3.9 Suppose $T\colon \mathbb{P}_3 \to \mathbb{R}^4$ is defined by

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a - b + 2d \\ 2a - c + d \\ -a + 2b - 2c + d \\ a - 2b + 2c - d \end{bmatrix}.$$

Then the matrix representation with respect to the standard bases for $\mathbb{P}_3$ and $\mathbb{R}^4$ is the familiar

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}.$$

From our previous example, we know

$$\text{Ker } A = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} x_4 \colon x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Now, to translate this to $\text{Ker } T$, we need to view these as coordinate vectors with respect to our standard basis for $\mathbb{P}_3$. Thus, $\text{Ker } T = \text{Span}\left\{1 + 3x + 3x^2 + x^3\right\}$.

Exploration 116 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix representation A for the linear transformation T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to find a basis for both $\text{Imag } T$ and $\text{Ker } T$.

Section Highlights

- Any column of a matrix, A, that has a pivot when row reduced is called a pivot column. Note that the pivot column is a column in the original matrix A, not its row reduced form. See Definition 4.3.1.
- A set of vectors in $\mathbb{R}^n$ is linearly independent if and only if the matrix with those vectors as columns has a pivot in every column. See Theorem 4.3.1.
- A set of vectors in $\mathbb{R}^n$ is a spanning set for $\mathbb{R}^n$ if and only if the matrix formed using those vectors as columns has a pivot in every row when row reduced. See Corollary 4.3.3.
- Suppose $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a subspace of $\mathbb{R}^m$. The set of pivot columns of $A = [\vec{v}_1 \ \ldots \ \vec{v}_n]$ is a basis for H. See Theorem 4.3.2.
- The set of pivot columns of a matrix, A, is a basis for $\text{Col } A$, which is the image of the linear transformation T_A. See Example 4.3.3.
- The coordinate vector for a vector $\vec{x}$ with respect to a basis $\mathcal{B}$ can be computed in the following manner:
  - Use the vectors in $\mathcal{B}$ as the columns of a matrix and augment it with $\vec{x}$.
  - Row reduce to reduced row-echelon form.
  - Read the coordinate vector for $\vec{x}$ from the augmented column.

  See Example 4.3.5 and Example 4.3.7
- Row operations do not change the kernel of a matrix. Thus, to find the kernel, row reduce to reduced row-echelon form; the parametric solution to the row reduce matrix augmented with the zero vector describes the kernel. See Example 4.3.8.

Exercises for Section 4.3

4.3.1. Determine whether the set is linearly dependent or linearly independent. If they are linearly dependent, set up a matrix and row reduce to find a way to write one of the vectors as a linear combination of the others.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix} \right\}$

(g) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

4.3.2. Use the concept of pivot columns to determine whether the set is a basis for $\mathbb{R}^3$.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

4.3.3. Use the concept of pivot columns to determine a basis for each subspace H.

(a) $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $H = \text{Span}\left\{1 + x, 2 - x + x^4, x^2 - 3x^3, 1 - 2x + x^4\right\}$

(d) $H = \text{Span}\left\{1 + x + x^4, 1 - 2x, 1 + x^2 - 3x^3, 2 - x + x^4\right\}$

(e) $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

4.3.4. Consider the basis $\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Write the vector $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of these basis vectors.

4.3.5. Consider the basis $\mathcal{B}_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Write the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of these basis vectors.

4.3.6. Consider the basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Find the coordinate vectors below.

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_{\mathcal{B}_2}$

(b) $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}_{\mathcal{B}_2}$

(c) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}_2}$

(d) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_{\mathcal{B}_2}$

4.3.7. Use the method from Example 4.3.6 to find the matrix for T with respect to the given basis $\mathcal{B}$.

(a) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ 3x_1 + 3x_2 \end{bmatrix} \text{ with respect to the basis } \mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ 3x_1 + 3x_2 + x_3 \end{bmatrix} \text{ with respect to the basis } \mathcal{B} = \left\{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

(c) $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ be defined by

$$T\left(a_0 + a_1 x + a_2 x^2\right) = (a_0 + a_1) + (a_2)x^2 \text{ with respect to the basis } \mathcal{B} = \left\{1, 1+x, 1+x+x^2\right\}$$

4.3.8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 - x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to compute $\mathrm{Imag}\ T$ and $\mathrm{Ker}\ T$.

4.3.9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to compute $\mathrm{Imag}\ T$ and $\mathrm{Ker}\ T$.

4.3.10. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be defined by

$$T\left(a_0 + a_1 x + a_2 x^2\right) = a_0 x + 3a_0 x^2 + (a_1 + a_2)x^3.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{P}_2$ and $\mathbb{P}_3$. Use this matrix to compute $\mathrm{Imag}\ T$ and $\mathrm{Ker}\ T$.

4.3.11. Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be defined by

$$T\left(a_0 + a_1 x + a_2 x^2\right) = \begin{bmatrix} a_0 + a_1 + a_2 \\ 2a_0 - a_2 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{P}_2$ and $\mathbb{R}^2$. Use this matrix to compute $\mathrm{Imag}\ T$ and $\mathrm{Ker}\ T$.

4.4 Matrix Operations

Where are we now?[20] We've introduced matrices as a convenient way to represent linear transformations. We've also developed a systematic way to simplify them, row operations and Gauss-Jordan Elimination, and seen that since this helps us solve systems of equations, it is essentially an alternative method for doing everything we've learned so far. Well, except showing a set's a vector space. Row operations are no help there. That does remind us though that we should talk about vector space-style operations on matrices. That's where we are.

20: As a civilization?

I think I have a map here somewhere. . .

Oh! The margin! We are in the margin.

Addition, Scalar Multiplication, and Matrix/Vector Multiplication

We've made a big deal so far about how a matrix can be used to represent a linear transformation. Now, a linear transformation is a function, and an operation we have for two functions with the same domain and codomain is function addition.

Example 4.4.1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 3$ and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x - 4$. There is then a function

$$f + g\colon \mathbb{R} \to \mathbb{R}$$

defined by

$$(f + g)(x) = f(x) + g(x) = (x + 3) + (x - 4) = 2x - 1.$$

This is likely something you've seen before. Now, we haven't discussed whether adding two linear transformations with the same domain and codomain should give us a new linear transformation. It does; we just haven't mentioned it before. We can also scale a linear transformation to obtain a new linear transformation. Maybe we should say this using official-sounding words.

Theorem 4.4.1 *Let $T\colon V \to W$ and $S\colon V \to W$ be linear transformations.*

(a) *Then $T + S\colon V \to W$ defined using the usual function addition is also a linear transformation.*

(b) *Let $k \in \mathbb{R}$. Then $kT\colon V \to W$ defined by scaling each output of the linear transformation T by k is also a linear transformation.*

The proof of this gets us a bit off-track, so we've put it in the exercises. You do remember how to show a function is a linear transformation, right?

Let's get back to talking about matrices. We've just defined two operations on linear transformations. Since we've made it quite clear that matrices and linear transformations are deeply linked, there should be analogous operations for matrices. We call these componentwise addition and scalar multiplication.

First, we can illustrate this with 2×2 matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \quad \text{and}$$
$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix},$$

for some scalar $k \in \mathbb{R}$. Generally, we have the following definition.

Definition 4.4.1
- *The sum of two $m \times n$ matrices*[21] *$[a_{ij}]$ and $[b_{ij}]$ is the $m \times n$ matrix $[c_{ij}]$ whose entries satisfy $c_{ij} = a_{ij} + b_{ij}$.*
- *The scalar multiple of a matrix $[a_{ij}]$ by a scalar $k \in \mathbb{R}$ is the $m \times n$ matrix $[d_{ij}]$ whose entries satisfy $d_{ij} = ka_{ij}$.*

21:  *They must have the same dimensions!*

Exploration 117 Use the definitions above to compute these:

- $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 1 \\ -1 & -2 & 4 \end{bmatrix} =$

- $3\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & -2 \end{bmatrix} =$

We've saved matrices for the latter half of the book, but we could have discussed them back in Section 1.1 because we can also think of them as vectors. For example, observe that

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \mathcal{M}_{2\times 3} \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \in \mathbb{R}^6$$

contain the same information. Our operations of vector addition and scalar multiplication in this case would even match this new componentwise addition and scalar multiplication for matrices. Thus, the following theorem shouldn't be a surprise.

Theorem 4.4.2 *The set of $m \times n$ matrices, denoted $\mathcal{M}_{m\times n}$, is a vector space with componentwise addition and scalar multiplication.*

We could further explore the identification between matrices in $\mathcal{M}_{m\times n}$ and vectors in $\mathbb{R}^{mn}$ to actually form an isomorphism and prove this theorem. Moveover, once we've agreed $\mathcal{M}_{m\times n}$ is a vector space, we can get the following corollary.

Corollary 4.4.3 *The set of linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a vector space with function addition and scalar multiplication.*

The statement follows because we could map any matrix A to its induced linear transformation T_A and get an isomorphism. This game of declaring something is a vector space because we can define an isomorphism seems like a lot of fun,[22] but let's get back to matrices.

22: Because it is!

While we didn't previously talk about how matrices form a vector space, we did define a way to multiply a matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$ and a vector $\vec{x} \in \mathbb{R}^n$:

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n.$$

We've gotten a lot of mileage out of this definition; most notably, it allowed us to think of matrices as functions, which paved the way for using matrices as representations for linear transformations. We should "jump start" something to complete the lap of car metaphors.[23]

23: or perhaps. . . we should tap on the brakes?

Oh, well done, Bubbles!

Matrix Multiplication

The next logical operation on matrices would be multiplication of two matrices.[24] As with vectors, there are a lot of ways to think about a product of matrices, but a particularly useful and instructive one comes from function composition, an essential tool in function theory.[25] We saw in Theorem 3.2.3 that the composition of two linear transformations is again a linear transformation.

24: Recall from Chapter 1 that this was complicated for vectors.

25: Recall from Chapter 3 that we needed function composition to make inverses.

Just as we just did with addition and scalar multiplication, we'd like to translate this operation to matrices. Suppose $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{m\times n}$ and $B = [\vec{b}_1 \cdots \vec{b}_p] \in \mathcal{M}_{n\times p}$; we have induced linear transformations $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^p \to \mathbb{R}^n$. From Theorem 3.2.3, we know the composition $T_A \circ T_B : \mathbb{R}^p \to \mathbb{R}^m$ forms a linear transformation. How can the matrices A and B to reflect this?

Let $\vec{x} \in \mathbb{R}^p$. Then we have

$$\begin{aligned} (T_A \circ T_B)(\vec{x}) &= T_A\left(T_B(\vec{x})\right) && \text{by definition of composition} \\ &= T_A(B\vec{x}) && \text{since } T_B(\vec{x}) = B\vec{x} \\ &= A(B\vec{x}) && \text{since } B\vec{x} \in \mathbb{R}^n \\ & && \text{and } T_A(\vec{u}) = A\vec{u} \text{ for all } \vec{u} \in \mathbb{R}^n. \end{aligned}$$

Thus, for composition to agree with matrix multiplication, we need to have our matrix product AB satisfy $(AB)\vec{x} = A(B\vec{x})$. Calculating the expression on the right hand side of this, we have

$$\begin{aligned} A(B\vec{x}) &= A\left(x_1\vec{b}_1 + \cdots x_n\vec{b}_n\right) \\ &= x_1A\vec{b}_1 + \cdots + x_nA\vec{b}_n \;=\; [A\vec{b}_1 \cdots A\vec{b}_n]\vec{x}. \end{aligned}$$

This then dictates precisely what our matrix product should be.

Definition 4.4.2 *Let $A \in \mathcal{M}_{m\times n}$ and $B = [\vec{b}_1 \cdots \vec{b}_p] \in \mathcal{M}_{n\times p}$. Then we define the* ***product of matrices*** *A* ***and*** *B to be the matrix $AB \in \mathcal{M}_{m\times p}$*

given by

$$AB = A[\vec{b}_1 \cdots \vec{b}_p] = [A\vec{b}_1 \cdots A\vec{b}_p].$$

Defining the product AB in this fashion gives us a natural way to have a matrix representation for a composition of linear transformations.

Theorem 4.4.4 *Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, and let $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T_B\colon \mathbb{R}^p \to \mathbb{R}^n$ be the induced linear transformations. Then the matrix $AB \in \mathcal{M}_{m\times p}$ has induced linear transformation $T_{AB} = T_A \circ T_B$.*

PROOF. Note first that the product AB makes sense based on the dimensions of the matrices A and B. This is equivalent to checking that the composition $T_A \circ T_B$ also maps between the appropriate spaces.[26] Let $\vec{x} \in \mathbb{R}^p$. Using the definition of matrix multiplication and our definition for the linear transformation induced by a matrix, we have $T_{AB}(\vec{x}) = (AB)\vec{x} = A(B\vec{x}) = A(T_B(\vec{x})) = T_A(T_B(\vec{x})) = T_A \circ T_B(\vec{x})$.[27] □

26: Exercise!

27: You will also justify each of these equalities as an exercise.

Example 4.4.2 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Let's calculate AB. To make it clear what's happening, let's also shade the second column of B.

$$AB = A\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} A\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & A\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 9\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 11\begin{bmatrix} 3 \\ 6 \end{bmatrix} & 8\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 12\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7+18+33 & 8+20+36 \\ 28+45+66 & 32+50+72 \end{bmatrix}$$

$$= \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}.$$

Let's calculate BA. Hey, three columns in A? We'll shade the middle column to help keep things clear.

$$BA = B\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} B\begin{bmatrix} 1 \\ 4 \end{bmatrix} & B\begin{bmatrix} 2 \\ 5 \end{bmatrix} & B\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} & \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} & \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} + 4\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} & 2\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} + 5\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} & 3\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} + 6\begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7+32 & 14+40 & 21+48 \\ 9+40 & 18+50 & 27+60 \\ 11+48 & 22+60 & 33+72 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 54 & 69 \\ 49 & 68 & 87 \\ 59 & 82 & 105 \end{bmatrix}.$$

You probably noticed that $AB \neq BA$. Lots of types of multiplication are commutative, but evidently, matrix multiplication is not. Matrix multiplication is so badly not commutative, the product AB and the product BA are not even the same dimension!

Exploration 118 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Calculate AB and BA and verify that matrix multiplication does not commute.

Matrix Multiplication via Transpose

There is another way to think about this matrix multiplication that is often used. We will first need a new definition.

Definition 4.4.3 *Let $A \in \mathcal{M}_{m\times n}$. The* ***transpose of*** *A, denoted A^T, is the matrix in $\mathcal{M}_{n\times m}$ derived from A by making the jth column of A into the jth row for each $1 \leq j \leq n$.*

Example 4.4.3 Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}.$$

Now that we have that, we make the following observation:

$$\mathcal{M}_{n\times 1} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \right\} = \mathbb{R}^n,$$

and

$$\mathcal{M}_{1\times n} = \{[x_1 \cdots x_n] : x_i \in \mathbb{R}\} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T : x_i \in \mathbb{R} \right\} \cong \mathbb{R}^n.$$

[28] Clearly, the concept of a matrix transpose applies on a single row or column vector because they are matrices. This gives us the following fun interpretation of matrix multiplication.

28:  I always write my vectors horizontally with $\langle , \rangle$ brackets, so it is *your* weird set that is isomorphic to $\mathbb{R}^n$.

Theorem 4.4.5 *Let $A \in \mathcal{M}_{m\times p}$ and $B \in \mathcal{M}_{p\times n}$. Then the product $AB \in \mathcal{M}_{m\times n}$ is the matrix whose entry in the ith row and jth column is the inner product of the transpose of the ith row of A with the jth column of B for all $1 \leq i \leq m$ and $1 \leq j \leq n$. That is,*

$$AB = [(ab)_{ij}], \quad \text{where} \quad (ab)_{ij} = \vec{r}_i^{\,T} \cdot \vec{b}_j,$$

where $\vec{r}_i$ is the ith row of A and $\vec{b}_j$ is the jth column of B.

Exploration 119 We will prove Theorem 4.4.5 together! First, let $\vec{x} \in \mathbb{R}^p$ and write out $A\vec{x}$ as a *single* vector

Now convince yourself that for any vector $\vec{x} \in \mathbb{R}^p$, we have

$$A\vec{x} = \begin{bmatrix} \vec{r}_1^{\,T} \cdot \vec{x} \\ \vdots \\ \vec{r}_m^{\,T} \cdot \vec{x} \end{bmatrix},$$

where $\vec{r}_i$ are the row vectors of A for $1 \leq i \leq m$.

Now use the fact that $AB = [A\vec{b}_1 \cdots A\vec{b}_n]$ to complete the proof.

Example 4.4.4 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Let's calculate AB using this new method. Note first that we have

$$\vec{r}_1^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{r}_2^T = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}, \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix}$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} \vec{r}_1^T \cdot \vec{b}_1 & \vec{r}_1^T \cdot \vec{b}_2 \\ \vec{r}_2^T \cdot \vec{b}_1 & \vec{r}_2^T \cdot \vec{b}_2 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \\ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}. \end{aligned}$$

Properties of Matrix Multiplication

Like any other respectable mathematical operation, matrix multiplication has some nice properties. We've already discovered the deplorable fact that matrix multiplication is not commutative, and that makes everybody very sad.[29] Before we celebrate some things that matrix multiplication actually does well, we need a definition:

29:  Except the algebraists specializing in non-commutative algebras.

Definition 4.4.4 *The* ***identity matrix*** *is the square matrix $I_n \in \mathcal{M}_{n\times n}$ whose columns are the standard basis for $\mathbb{R}^n$ in order. That is,*

$$I_n = [\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Note that the identity matrix corresponds to the identity map for a vector space, the linear transformation that sends each vector to itself. Formally, $T_I = T$, where $T\colon V \to V$ is given by $T(\vec{x}) = \vec{x}$ for all $\vec{x} \in V$. Now, commence celebration!

Theorem 4.4.6 *Let $k \in \mathbb{R}$, and $A \in \mathcal{M}_{m\times n}$, and let B and C be matrices of the size necessary for each of the following operations to be well-defined. Then*

(a) $(AB)C = A(BC)$ *(associativity of matrix multiplication)*
(b) $A(B+C) = AB+AC$ *(left distribution of matrix multiplication)*
(c) $(B+C)A = BA+CA$ *(right distribution of matrix multiplication)*
(d) $k(AB) = (kA)B = A(kB)$
(e) $I_mA = A = AI_n$ *(identity for matrix multiplication)*

Those look wonderful! Unfortunately, their proofs do not. They are not difficult, but the amount of notation and page space required makes them an eyesore we would rather avoid. Let's verify one of these for 2×2 matrices just so we don't feel quite so bad about cheating you out of the glorious satisfaction of a thorough proof.[30]

30: I always knew you authors were dirty cheaters! How dare you not include the proof?! I know Nicky, for one, will never forgive you.

Umm...I know the proof here, and I'm actually good with the fact that they chose not to show it.

Really? I'm not. I demand satisfaction!

Example 4.4.5 Let's start with some 2×2 matrices.

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

Now, we will verify $A(B+C) = AB+AC$. This one should give us a nice flavor for what all of the proofs would have looked like.

$$\begin{aligned}
A(B+C) &= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1(b_1+c_1)+a_2(b_3+c_3) & a_1(b_2+c_2)+a_2(b_4+c_4) \\ a_3(b_1+c_1)+a_4(b_3+c_3) & a_3(b_2+c_2)+a_4(b_4+c_4) \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1+a_1c_1+a_2b_3+a_2c_3 & a_1b_2+a_1c_2+a_2b_4+a_2c_4 \\ a_3b_1+a_3c_1+a_4b_3+a_4c_3 & a_3b_2+a_3c_2+a_4b_4+a_4c_4 \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_3b_1+a_4b_3 & a_3b_2+a_4b_4 \end{bmatrix} + \begin{bmatrix} a_1c_1+a_2c_3 & a_1c_2+a_2c_4 \\ a_3c_1+a_4c_3 & a_3c_2+a_4c_4 \end{bmatrix} \\
&= AB+AC
\end{aligned}$$

Now that you've seen Example 4.4.5, you can imagine how the general proofs for these statements go.[31]

31: I take back that dirty cheater comment. Thank you for omitting these.

There's something cool we get as a consequence of these properties.

Corollary 4.4.7 *Let $A \in \mathcal{M}_{n\times m}$. There is a linear transformation*

$$M_A\colon \mathcal{M}_{m\times p} \to \mathcal{M}_{n\times p}$$

defined by $M_A(B) = AB$ for any $B \in \mathcal{M}_{m\times p}$.

The proof of this is a wonderful application of Theorem 4.4.6 that you should expect to see as an exercise.

Properties of the Transpose

As part of our second definition of matrix multiplication, we introduced the concept of the transpose of a matrix. This concept will actually have some nice connections to applications, so we should say a bit more about it. Recall that computing the transpose of a matrix is done by taking a row vector and making it a column vector.[32] There are several nice properties of this operation.

32:  Or vice-versa, right?

Theorem 4.4.8 *Let* $A, B \in \mathcal{M}_{m\times n}$ *and* $\alpha \in \mathbb{R}$. *Then*
(a) $(A+B)^T = A^T + B^T$,
(b) $(\alpha A)^T = \alpha A^T$, *and*
(c) $(A^T)^T = A$.

PROOF. Some of these should be done as an exercise, but we'll get the ball rolling by doing a together. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ for $1 \le i \le n$ and $1 \le j \le m$. Then, $A + B = [a_{ij} + b_{ij}]$ by the definition of addition of matrices. Note that we defined the transpose as swapping rows and columns, but we can also say what happens to each entry. If a_{ij} is in the ith row and jth column of A, then it is in the jth row and ith column of A^T. Let's denote this entry in A^T as $\overline{a}_{ji}$ and note that it is equal to a_{ij}. Keeping this notational convention, we can denote the entry in the jth row and ith column of $(A+B)^T$ as $\overline{a+b}_{ji}$ and it will be equal to $a_{ij}+b_{ij}$. This is then equal to $\overline{a}_{ji}+\overline{b}_{ji}$. Thus, $(A+B)^T = A^T + B^T$. $\square$

This next one is a fun property.

Theorem 4.4.9 *Let* $A \in \mathcal{M}_{m\times p}$ *and* $B \in \mathcal{M}_{p\times n}$. *Then*
$$(AB)^T = B^T A^T.$$

PROOF. It is easy to check that both $(AB)^T, B^TA^T \in \mathcal{M}_{n\times n}$. Let c_{ij} be the entry in B^TA^T in the ith row and jth column. By Theorem 4.4.5, c_{ij} is the inner product of the ith row of B^T and the jth column of A^T. This is the same as the inner product of the j row of A and the ith column of B. It follows that c_{ij} is also the entry in $(AB)^T$ in the ith row and jth column. $\square$

Row Operations as Matrix Multiplication

We've now discussed several matrix operations and seen how they are tied closely to analogous operations on linear transformations. One we haven't yet connected back to linear transformations is row operations, so let's do that now. Let $A \in \mathcal{M}_{n\times m}$. Since each of our row operations on A treat an entire row of A the same, there is actually a linear transformation $T_r\colon \mathbb{R}^n \to \mathbb{R}^n$ for each of the row operations r. For example, the row operation that swaps the first two rows of A corresponds to the linear transformation $T_{\vec{r}_1 \leftrightarrow \vec{r}_2}\colon \mathbb{R}^n \to$

$\mathbb{R}^n$ defined as

$$T_{\vec{r}_1 \leftrightarrow \vec{r}_2}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Swapping any other rows would be similar. The row operation that scales the first row of A by a nonzero scalar $k \in \mathbb{R}$ corresponds to the linear transformation $T_{k\vec{r}_1 \to \vec{r}_1} \colon \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$T_{k\vec{r}_1 \to \vec{r}_1}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} kx_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Scaling any other rows would be similar. The row operation that replaces the first row with the sum of the first row and the second row scaled by $k \in \mathbb{R}$ corresponds to the linear transformation $T_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1} \colon \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$T_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 + kx_2 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

As before, doing this with other rows would be similar. Each of these linear transformations then has a matrix representation E_r with respect to the standard basis.

$$E_{\vec{r}_1 \leftrightarrow \vec{r}_2} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad E_{k\vec{r}_1 \to \vec{r}_1} = \begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$E_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1} = \begin{bmatrix} 1 & k & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Now, we didn't actually prove that these were linear transformations or that these were the matrix representations.[33] Since this section is already getting a bit long, we'll relegate these formalities to the exercises.

We defined these to correspond to row operations, but each linear transformation, $E_r \colon \mathbb{R}^n \to \mathbb{R}^n$, will really only convert a column at a time. However, using our definition of matrix multiplication and Theorem 4.4.7, we can define $E_r \colon \mathcal{M}_{n\times m} \to \mathcal{M}_{n\times m}$ by

$$E_r A = E_r[\vec{a}_1\ \vec{a}_2\ \cdots \vec{a}_m] = [E_r\vec{a}_1\ E_r\vec{a}_2\ \cdots E_r\vec{a}_m].$$

This means we can row reduce A using matrix multiplication! These matrices that correspond to row operations are important enough to have their own name.

33:  I believe that these are linear transformations. Seems pretty obvious.

 Didn't you demand satisfaction for matrix addition just a couple of pages ago?

Definition 4.4.5 *We call $E \in \mathcal{M}_{n\times n}$ an* **elementary matrix** *if for any $A \in \mathcal{M}_{n\times m}$, the matrix EA is the matrix A after performing a row operation on A.*

If we look at the specific elementary matrices we've seen so far, they are all one row operation away from the identity matrix I_n.

Theorem 4.4.10 *If $B \in \mathcal{M}_{n\times m}$ is the result of performing a row operation on a matrix $A \in \mathcal{M}_{n\times m}$ and E is the result of performing that same row operation on I_n, then $B = EA$.*

This is very convenient. If you want to do a row operation to A, then you could just do it to I_n and multiply that with A, and you get the same result as if you did the row operation to A. The proof of this theorem has a very similar flavor to how we found the E_r above, so we will omit it.[34]

34:  CHEATERS!

Example 4.4.6 Let's see it in action though. Let us define

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Suppose we would like to swap the first and third rows of A. Then the appropriate elementary matrix E would be

$$E = E_{\vec{r}_1 \leftrightarrow \vec{r}_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This is exactly the identity matrix with the first and third rows swapped. Now,

$$EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & -2 \\ 2 & 1 & 0 \end{bmatrix}.$$

Exploration 120 Consider the matrices A and B below.

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 3 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -9 & -8 & 0 \end{bmatrix}$$

Find the elementary matrix E such that $B = EA$.

Section Highlights

- Matrices of the same size can be added. Combining this with scalar multiplication of matrices gives us the fact that the space of matrices of a given size forms a vector space. See Theorem 4.4.2.
- A matrix of size $m \times n$ and a matrix of size $n \times k$ can be multiplied to form a matrix of size $m \times k$. This multiplication is the matrix

equivalent to function composition for linear transformations. See Definition 4.4.2 and Theorem 4.4.4.

- Since function composition is not commutative, matrix multiplication is not commutative. See Example 4.4.2.
- The $n \times n$ matrix, I_n, that has 1's on the diagonal and 0's everywhere else is called the identity matrix. Multiplying a matrix by I_n does not change the matrix. See Definition 4.4.4.
- The transpose of an $m \times n$ matrix is the $n \times m$ matrix formed by turning rows into columns (or vice versa). See Definition 4.4.3.
- Elementary matrices are matrix representations of row operations, providing a way to do row operations as matrix multiplication. See Definition 4.4.5 and Theorem 4.4.10

Exercises for Section 4.4

4.4.1. Perform the indicated operations.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(b) $5\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

(d) $3\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

(e) $3\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}^T - 2\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}^T$

4.4.2. Multiply these matrices.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 1 \\ 2 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}^T\begin{bmatrix} 0 & 4 \\ 2 & 1 \\ 1 & -2 \end{bmatrix}^T$

(g) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix}^T\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}^T$

4.4.3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}, \qquad C = \begin{bmatrix} -7 & 8 \\ 9 & -10 \\ -11 & 12 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \qquad E = \begin{bmatrix} 7 \\ 9 \\ -10 \end{bmatrix}, \qquad F = \begin{bmatrix} -8 & 10 & 12 \end{bmatrix}$$

Compute the following matrices if it is possible. If it is not possible, draw your best dragon.

(a) AB

(b) BA

(c) $C(BD)$

(d) $B(CD)$

(e) EF

(f) FE

(g) $B(AE)$

(h) CE

(i) $(FC)D$

(j) $(DF)C$

4.4.4. Row reduce the matrices to reduced row-echelon form and find the elementary matrices for each row operation.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

4.4.5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 5x_2 + x_3 \\ x_3 \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}.$$

(a) Find the matrix representation A for T and the matrix representation B for S with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$.

(b) Compute AB and BA.

(c) Verify that the matrix for $T \circ S$ with respect to the standard basis for $\mathbb{R}^2$ is AB.

(d) Verify that the matrix for $S \circ T$ with respect to the standard basis for $\mathbb{R}^3$ is BA.

4.4.6. Verify the associative property for matrix multiplication for 2×2 matrices. That is, show

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\begin{bmatrix} x & y \\ u & w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} x & y \\ u & w \end{bmatrix}\right)$$

4.4.7. Verify the left distribution of matrix multiplication for 2×2 matrices.

4.4.8. Verify Theorem 4.4.9 for 2×2 matrices.

4.4.9. Prove Theorem 4.4.1. That is, verify these maps are linear transformations.

4.4.10. Prove M_A from Corollary 4.4.7 is a linear transformation.

4.4.11. Complete the proof of Theorem 4.4.8. That is, let $A \in \mathcal{M}_{m\times n}$ and $\alpha \in \mathbb{R}$.

(a) Prove $(\alpha A)^T = \alpha A^T$.

(b) Prove $(A^T)^T = A$.

4.4.12. Prove the following are linear transformations and verify the matrix representations with respect to the standard basis as stated in the text.

(a) $T_{\vec{r}_1 \leftrightarrow \vec{r}_2}$

(b) $T_{k\vec{r}_1 \to \vec{r}_1}$

(c) $T_{\vec{r}_1 + k\vec{r}_2 \to \vec{r}_1}$

4.5 Invertible Matrices

In the previous section, we learned about matrix multiplication and how it strengthens the connection between matrices and linear transformations. Well, when you hear multiplication, often you think about division as well. Really though, you are thinking of a way to "undo" the multiplication.[35] While we do not have a concept of division for matrices, we do have an "inverse" of a matrix. This will match what you would expect from our discussion of inverses of functions from Section 3.1 because of the connection between matrix multiplication and composition of linear transformations, and our row operations will give us a nice computational algorithm to compute these inverses when they exist. We've given away enough of what's coming; we should just get started.

35: I was told that subtraction and division are a lie and that fields only have two operations, usually thought of as addition and multiplication.

Wha? Ricky?

That's right. Subtraction and division are just the inverse operations of addition and multiplication, respectively.

Oh. You.

Matrix Inverses from Linear Transformations

Matrix representations have already proven to be a very convenient tool for understanding pretty much everything about linear transformations, so it should come as little surprise that they also simplify determining invertibility and finding inverses for linear transformations as well.[36] Recall Definition 3.1.4; for vector spaces V and W and the linear transformation $T\colon V \to W$, we call $S\colon W \to V$ the inverse of T if

- for any $\vec{v} \in V$, $(S \circ T)(\vec{v}) = \vec{v}$ and
- for any $\vec{w} \in W$, $(T \circ S)(\vec{w}) = \vec{w}$.

36: Is there an isomorphism between matrix stuff and linear transformation stuff?

No. Maybe a one-to-one and onto function, though. You'd still probably have to be quite finicky about how you define your domain and codomain.

Exercise!

Moreover, we know from Theorem 3.1.3 that T is invertible if and only if it is both one-to-one and onto; thus, a linear transformation is invertible if and only if it is an isomorphism. Since T is an isomorphism, we also know that V and W are isomorphic, so by Corollary 3.3.6, they must have the same dimension.

If A and B are matrix representations of T and S respectively, then we know $A, B \in \mathcal{M}_{n\times n}$ for some n since V and W are the same dimension.[37] To keep things from being too crazy, let's restrict these matrix representations to be relative to a specific chosen basis for V and also a specific chosen basis for W. In this context, we see that the criteria for invertibility above becomes the following. For the linear transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^n$, we call $S_B\colon \mathbb{R}^n \to \mathbb{R}^n$ the inverse of T_A if for any $\vec{x} \in \mathbb{R}^n$,

$$(S_B \circ T_A)(\vec{x}) = BA\vec{x} = \vec{x} = AB\vec{x} = (T_A \circ S_B)(\vec{x}).$$

37: Did you catch that these are square matrices?

Wait, did we define a square matrix anywhere?

You're asking me? In case we didn't, it's just a matrix with the same number of rows and columns.

From Definition 4.4.4, we already know the identity matrix I_n is a matrix such that $I_n\vec{x} = \vec{x}$ for any $\vec{x} \in \mathbb{R}^n$. That's not all. . . well, actually, that is all.

Theorem 4.5.1 *The identity matrix $I_n \in \mathcal{M}_{n\times n}$ is the unique matrix such that for all $\vec{x} \in \mathbb{R}^n$, $I_n\vec{x} = \vec{x}$.*

PROOF. Suppose there is some matrix $A = [\vec{a}_1 \cdots \vec{a}_n] \in \mathcal{M}_{n\times n}$ such that for all $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = \vec{x}$. We also know that for all $\vec{x} \in \mathbb{R}^n$, $I_n\vec{x} = \vec{x}$, so for all $\vec{x} \in \mathbb{R}^n$, we have

$$I_n\vec{x} = A\vec{x}.$$

In particular, for any $1 \leq j \leq n$, we have $\vec{e}_i = I_n \vec{e}_i = A\vec{e}_i = \vec{a}_i$. Thus, the columns of A are the vectors $\vec{e}_1, \ldots, \vec{e}_n$, so $A = I_n$. □

Thus, the linear transformation T_A above is invertible if and only if there is another linear transformation S_B such that $AB = I_n = BA$. This motivates the following definition.

Definition 4.5.1 *A matrix $A \in \mathcal{M}_{n\times n}$ is* invertible *if there is another matrix $B \in \mathcal{M}_{n\times n}$ such that*

$$AB = I_n = BA.$$

We call the matrix B the inverse *of the matrix A and denote it as A^{-1}.*

Example 4.5.1 Here are a few matrices. Let's see if they are inverses of each other.

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix}.$$

It's easy to check that $AC = CA = I_2$, but $AB \neq I_2$ and $BC \neq I_2$. Thus, C is the inverse of A (and vice versa). That is, $C = A^{-1}$ and $A = C^{-1}$.

Note that this definition is stated just in terms of matrices and matrix multiplication. The concept of an inverse here is really "undoing" matrix multiplication, so it's our form of "division." Note that it only works for a square matrix, and even then, there will be many square matrices that are not invertible. Just like we can't divide by 0, there will be some square matrices with no inverses.[38] By the discussion prior to the definition and Theorem 4.4.4, we then have the following theorem.

38: Do you hear Jimmy Buffett music? I hear Jimmy Buffet suddenly for some reason.

What's he singing?

…something about boats?

Nope. It's just you.

Theorem 4.5.2 *A matrix $A \in \mathcal{M}_{n\times n}$ is invertible if and only if the induced linear transformation T_A is invertible.*

That's great! To tell whether a matrix is invertible, we just need to check whether the induced linear transformation is both one-to-one and onto. Maybe it'd be better, though, if we could tell this from the matrix itself. Let's revisit these concepts now that we know more about matrices.

One-to-one and Onto Using Pivots

In Chapter 3, we learned that a function is one-to-one if every element in the domain is mapped to a distinct output in the codomain. For linear transformations, Theorem 3.3.1 told us that a linear transformation will be one-to-one if and only if its kernel is just the zero vector. We can restate this now using a matrix equation to better align with our recent discussions.

Theorem 4.5.3 *A linear transformation $T\colon V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is one-to-one if and only if the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution, $\vec{x} = \vec{0}$.*

Proof. This follows immediately from Theorem 3.3.1 and the definition of $\operatorname{Ker} A$. □

We also saw in Theorem 3.5.5 that a linear transformation will be one-to-one if and only if the columns of the matrix representation are linearly independent. We can now use row reduction to determine this, so we can make the following statement.

Corollary 4.5.4 *A linear transformation $T: V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is one-to-one if and only if every column of A is a pivot column.*

PROOF. This is a restatement of Theorem 3.5.5 using Theorem 4.3.1. ☐

Exploration 121 Suppose the matrices below are matrices corresponding to linear transformations. Which ones correspond to a linear transformation that is one-to-one? Circle them.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

That was so effective! It turns out detecting whether a linear transformation is onto can be similarly efficient. We learned in Chapter 3 that a function is onto when everything in the codomain appears as an output for some input from the domain. For linear transformations, we saw that the desired version is that the image is equal to the codomain. We shall restate the definition of onto in the context of matrix equations first.

Corollary 4.5.5 *A linear transformation $T: V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is onto if and only if for every $\vec{b} \in \mathbb{R}^m$, the matrix equation $A\vec{x} = \vec{b}$ has a solution.*

PROOF. This follows immediately from the definition of onto, Theorem 3.5.2, and Theorem 4.1.1. ☐

We also saw in Theorem 3.5.5 that a linear transformation will be onto if the columns of the matrix representation span the codomain. From Theorem 4.2.8, we know the dimension of the image will be equal to the number of pivot columns. The dimension of the codomain is equal to the number of rows, so for a linear transformation to be onto, we need the number of rows to be equal to the number of pivots. Since each row can have only one pivot, we can state the following theorem.

Theorem 4.5.6 *A linear transformation $T: V \to W$ with matrix representation $A \in \mathcal{M}_{m\times n}$ is onto if and only if A can be row reduced to have a pivot in every row.*

PROOF. Read the paragraph before the theorem. That's the proof. ☐

Exploration 122 Suppose the matrices below are matrices corresponding to linear transformations. Which ones correspond to a linear transformation that is onto? Circle them.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now that we've talked about both one-to-one and onto, what about matrices for linear transformations that are both? We know for this we need to only consider square matrices. In this case, a pivot in every column is equivalent to a pivot in every row. This agrees with our expectations from Theorem 3.3.11 that a linear transformation between spaces of equal dimension will be either both one-to-one and onto or neither. This also gives us a nice condition for when a linear transformation is invertible.

Corollary 4.5.7 *A linear transformation $T\colon V \to V$ with matrix representation $A \in \mathcal{M}_{n\times n}$ is invertible if and only if A has n pivots.*

Computing the Inverse

When dealing with 2×2 matrices, there is a convenient formula for the inverse when it exists. This formula can be computed using a bit of algebra, and we've included just such a computation for you in the exercises. For now, we'll just tell you the answer though.

Theorem 4.5.8 *Given the matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathcal{M}_{2\times 2}$, the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc \neq 0$.

Exploration 123 Use Theorem 4.5.8 to find A^{-1} for

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Now we know how to find the inverse of a 2×2 matrix, but what about other for larger matrices? Well, from Theorem 4.5.2 and the discussion preceding it, the inverse of a matrix A is the matrix for the inverse linear transformation T_A^{-1}. Now, how do we find this matrix? Coordinate vectors!

Example 4.5.2 Let's use a 2×2 matrix so that we can check our work with the formula from Theorem 4.5.8. Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}.$$

Now, with the goal of understanding T_A^{-1}, let's begin by finding where T_A maps the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$.

$$T_A(\vec{e}_1) = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T_A(\vec{e}_2) = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Note that we just recovered the column vectors from A here! This is always what happens when we input the standard basis vectors to our linear transformation. Also, we can see that

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^2$, so T_A is an isomorphism and A is invertible. Now, we know $T_A^{-1} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that

$$T_A^{-1}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.27}$$

$$T_A^{-1}\left(\begin{bmatrix} -1 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.28}$$

Now, to find the matrix for T_A^{-1}, we need to determine $T_A^{-1}(\vec{e}_1)$ and $T_A^{-1}(\vec{e}_2)$, but we only know what $T_A^{-1}(\vec{b}_1)$ and $T_A^{-1}(\vec{b}_2)$ are from Equations 4.27 and 4.28. If only there was a way to write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\vec{b}_1$ and $\vec{b}_2$... wait! There is! For this, we must find the coordinate vectors for $\vec{e}_1$ and $\vec{e}_2$ with respect to the basis $\mathcal{B}$. We can do this with row reduction!

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

This says that $\vec{e}_1 = 3\vec{b}_1 + 2\vec{b}_2$ and $\vec{e}_2 = -\vec{b}_1 - \vec{b}_2$. Most important for our purposes though, we have that $T_A^{-1}(\vec{e}_1) = 3T_A^{-1}(\vec{b}_1) + 2T_A^{-1}(\vec{b}_2) = 3\vec{e}_1 + 2\vec{e}_2$ and $T_A^{-1}(\vec{e}_2) = -T_A^{-1}(\vec{b}_1) - T_A^{-1}(\vec{b}_2) = -\vec{e}_1 - \vec{e}_2$. That is,

$$T_A^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T_A^{-1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Thus, the matrix representation for T_A^{-1} with respect to the standard basis of $\mathbb{R}^2$ is

$$B = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}.$$

Note that this was exactly the right side of the augmented matrix we used for our row reduction. Because we are using the standard basis here and the matrix we started with was invertible, it will always work out like this. If the matrix we started with had not been invertible, we would have still gotten

column vectors as our outputs when we input the standard basis vectors to T_A, but these would not have formed a basis.
This method using row reduction gives us an algorithm for both determining invertibility and computing the inverse of the matrix when it exists.

Exploration 124 Use the formula for the inverse of a 2×2 matrix from Theorem 4.5.8 to check our answer from Example 4.5.2.

Theorem 4.5.9 *Suppose $A \in \mathcal{M}_{n\times n}$ is an invertible matrix. Then the augmented matrix $[A|I_n]$ row reduces to $[I_n|A^{-1}]$.*

The proof for this is essentially replacing the specific 2×2 matrix in Example 4.5.2 with a general $n \times n$ matrix. We'll save that for the Appendix.[39] There is also an alternative way to think about computing the inverse, but the outcome is this same algorithm. Note that we row reduced A to the identity matrix as part of Theorem4.5.9. There's a corollary there.

39: You can keep your row reduction proof.

Corollary 4.5.10 *A matrix A is invertible if and only if it is row equivalent to an identity matrix I_n for some positive integer n.*

Therefore, if a matrix $A \in \mathcal{M}_{n\times n}$ is invertible, there is a sequence of row operations $\{r_1, r_2, \ldots, r_k\}$ sending A to I_n. Converting this into elementary matrices gives us that

$$I_n = E_{r_k}(\cdots(E_{r_2}(E_{r_1}A) = (E_{r_k}\cdots E_{r_2}E_{r_1})A.$$

Thus, another way to find A^{-1} is to compute $(E_{r_k}\cdots E_{r_2}E_{r_1})$. Via Theorem 4.4.10, each of these matrices is obtained by applying the corresponding row operation to the identity matrix I_n. Thus, to compute A^{-1} with this method, we start with the identity matrix and then perform the same row operations as we used to row reduce A, in the same order. Thus, we can view Theorem 4.5.9 as keeping track of our elementary matrices with the augmented portion of the matrix.

Now that we've explained why our algorithm for computing the inverse works in two different ways, let's actually see it in action.

Example 4.5.3 Here's a matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Let's see if we can find an inverse. Well, after row reducing, we have

$$A \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so A is not invertible. Alright. Let's try

$$B = \begin{bmatrix} 1 & -2 & 6 \\ 1 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Good. B is invertible. Oh, drat! We forgot to augment B with the identity matrix. Well, at least it was easy to row reduce. Let's augment B with I_3 and try again.

$$[B \,|\, I_3] = \left[\begin{array}{ccc|ccc} 1 & -2 & 6 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -14 \\ 0 & 1 & 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus, $B^{-1} = \begin{bmatrix} 3 & 2 & -14 \\ 1 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

Exploration 125 Now it's your turn! Find the inverse of the matrix below using the method of augmentation by I_3.

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Nice job! Now, check your answer by multiplying the original matrix by your new suspected inverse. Did you get I_3? If not, you might need to check your row reduction.

Matrix Inverses and Equations

Maybe now is a good time to think about why we should care about the inverse of a matrix. The inverse of a matrix will eventually turn out to be useful in a lot of contexts, but the definition alone immediately provides a great tool. To solve the equation $2x = 4$ for x, we know we should "divide both sides by 2." However, what we're really doing is *multiplying both sides of the equation by the multiplicative inverse of* 2.[40] Now suppose we have a matrix equation $A\vec{x} = \vec{b}$. If A is invertible, we can employ the exact same strategy. If B is the

40:  See! I told you!

inverse of A, then we have $AB = BA = I_2$, so multiplying both sides of our matrix equation by B, we have

$$BA\vec{x} = B\vec{b},$$

and

$$\vec{x} = I_2\vec{x} = BA\vec{x} = B\vec{b}.$$

We can solve for $\vec{x}$ by multiplying by the inverse of A. This can be very convenient.

Exploration 126 Suppose the inverse of $A \in \mathcal{M}_{2\times 2}$ is

$$B = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solve the matrix equation

$$A\vec{x} = \begin{bmatrix} 42 \\ -12 \end{bmatrix}.$$

Section Highlights

- A matrix A is invertible if and only if its induced linear transformation T_A is invertible. This also means there is an inverse matrix A^{-1} such that
$$AA^{-1} = I_n = A^{-1}A.$$
See Definition 4.5.1 and Theorem 4.5.2.

- The inverse of an invertible matrix A is computed by augmenting A with an appropriately sized identity matrix I_n and row reducing to reduced row-echelon form. The inverse is the resulting augmented side. See Theorem 4.5.9.

- A linear transformation is onto if and only if the row reduced form of any matrix representing it has a pivot in every row. See Theorem 4.5.6.

- A linear transformation is one-to-one if and only if the row reduced form of any matrix representing it has a pivot in every column. See Corollary 4.5.4.

Exercises for Section 4.5

4.5.1. For each matrix, row reduce to determine whether the corresponding linear transformation is one-to-one, onto, both or neither.

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

4.5.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_3 \\ x_3 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to determine whether T is one-to-one, onto, both or neither.

4.5.3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_2 \end{bmatrix}.$$

Find the matrix A that represents T on the standard basis vectors for $\mathbb{R}^3$. Use this matrix to determine whether T is one-to-one, onto, both or neither.

4.5.4. Determine whether the following matrices are invertible. If the matrix is invertible, find the inverse.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -4 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}$

4.5.5. Let $A = \begin{bmatrix} 3 & 2 & 6 \\ -2 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and

$$\vec{b}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \vec{b}_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}.$$

Find the inverse of A and use it to solve $A\vec{x} = \vec{b}_1$, $A\vec{x} = \vec{b}_2$, and $A\vec{x} = \vec{b}_3$.

4.5.6. Find a matrix representation for the linear transformation to determine whether it is invertible.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 \\ x_1 + x_2 \end{bmatrix}$$

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}$$

(d) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix}$$

4.5.7. Let's see if we can find the general formula for the inverse of a 2×2 matrix using just matrices. Given a matrix $A \in \mathcal{M}_{2\times 2}$, we want to find a formula for a matrix B such that $AB = BA = I_2$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

We need to solve for e, f, g, and h in terms of a, b, c and d so that $AB = BA = I_2$.

(a) First, calculate AB.

(b) We want this result to be I_2, so use the fact that we should have 0 in two of the four components to solve for f and g in terms of the other constants.

(c) Substitute these expressions for f and g into the two expressions that should be equal to 1 and solve for e and h.

(d) Now you can substitute your expressions for e and h back into your expressions for f and g. You should now have e, f, g, and h in terms of a, b, c and d. A little more algebra should yield a nice formula for you.

4.6 Matrix Theorems

In this section, we've combined several nice results about matrices. The main work for these results has already occurred, so the goal here is primarily organize them and to remind you of them.

Subspaces Induced by Matrix Representations

First, let's introduce a new subspace related to a matrix. It shouldn't be surprising though since we know that each row is a vector.

Definition 4.6.1 *For a matrix $A \in \mathcal{M}_{m\times n}$, let $\vec{r}_i$ be the vector formed from the ith row of A for each $1 \le i \le m$. The* ***row space of*** *A, denoted $\text{Row }A$, is the span of these row vectors. That is,*

$$\text{Row }A = \text{Span}\{\vec{r}_1, \ldots, \vec{r}_m\}.$$

Theorem 4.6.1 *For a matrix $A \in \mathcal{M}_{m\times n}$, $\text{Row }A$ is a subspace of $\mathbb{R}^n$.*

PROOF. This follows from Theorem 3.6.3 by taking the transpose of your matrix. □

Now for something really cool. Suppose we have a linear transformation between two inner product spaces. We have fun subspaces of domains and codomains for linear transformations (the kernel and image, respectively), but what about the rest of the domain and codomain? You would not be shocked to find that the orthogonal complement of the kernel is a subspace of the domain,[41] and the orthogonal complement of the image is a subspace of the codomain (Theorem 2.4.2). What is surprising is that these orthogonal complements are also given by the matrix representation for the linear transformation. Behold!

41: The orthogonal complement of the kernel is even isomorphic to the image from a theorem in Section 3.3.

Theorem 4.6.2 *Suppose V and W are inner product spaces. Let $T : V \to W$ be a linear transformation represented by the $n \times m$ matrix A. Then*

$$\text{Ker }A = (\text{Row }A)^\perp \quad \textit{and} \quad \text{Ker }A^T = (\text{Col }A)^\perp.$$

PROOF. Let $A = [a_{ij}]$ for $1 \le i \le m$ and $1 \le j \le n$, let $\vec{r}_i$ for $1 \le i \le m$ be the row vectors of A, and let $\vec{a}_j$ for $1 \le j \le n$ be the column vectors of A. Then

$$\begin{aligned}
& \vec{x} \in \text{Ker }A \\
\Leftrightarrow\ & A\vec{x} = \vec{0} \\
\Leftrightarrow\ & x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0} \\
\Leftrightarrow\ & x_1\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1}\end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn}\end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0\end{bmatrix} \\
\Leftrightarrow\ & x_1a_{i1} + \cdots + x_na_{in} = 0 \text{ for } 1 \le i \le n \\
\Leftrightarrow\ & \vec{x}\cdot\vec{r}_i = 0 \text{ for } 1 \le i \le n.
\end{aligned}$$

Thus, $\vec{x} \in \text{Ker } A$ if and only if it is orthogonal to every row of A. Since $\text{Row } A$ is the span of the rows of A, the result follows. The other equality is then achieved by noting $\text{Row } A^T = \text{Col } A$. □

Exploration 127 Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$

Find $\text{Ker } A$ and check that it is orthogonal to each row vector.

Find $\text{Ker } A^T$ and check that it is orthogonal to each column vector.

Corollary 4.6.3 *Let A be an $n \times m$ matrix with induced linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$. Then*

$$\text{dom}\,(T_A) = \text{Ker } A \oplus \text{Row } A \quad \textit{and} \quad \text{codom}\,(T_A) = \text{Col } A \oplus \text{Ker } A^T.$$

We are now able to complete the vector spaces in our big commuting diagram; see Figure 4.3.

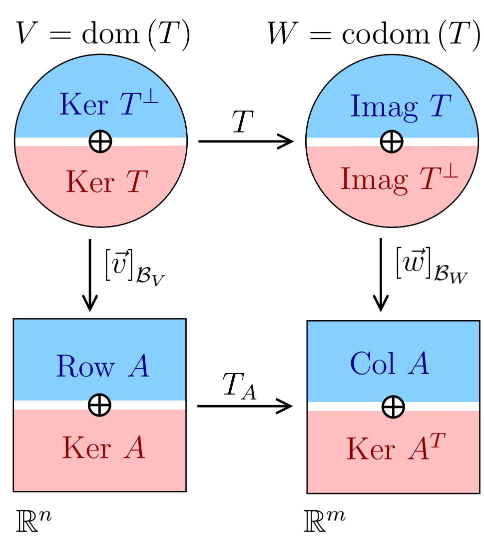

FIGURE 4.3. Some people refer to this as "the splits" of $\text{dom}\,(T_A)$ and $\text{codom}\,(T_A)$.

Theorem 4.6.4 (Invertible Matrix Theorem) *Let $A \in \mathcal{M}_{n\times n}$, and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A\vec{x}$. Note that the matrix A is a square matrix, one where the rows and columns have the same length. The following are equivalent statements.*

(a) *A is invertible.*

(b) *A can be row reduced to I_n.*

(c) *T is invertible.*

(d) *T is one-to-one.*

(e) *T is onto.*

(f) *A has a pivot in every column.*

(g) *A has a pivot in every row.*

(h) *The columns of A are linearly independent.*

(i) *The rows of A are linearly independent.*

(j) $\text{Ker } A = \{\vec{0}\}$.

(k) *A^T is invertible.*

PROOF. Let's actually take it from the top and bottom here. We know from Theorem 4.4.9 that $(AB)^T = B^T A^T$. Now, A is invertible if and only if there is some matrix B such that $AB = I_n$. Combining this with the fact that the identity matrix is its own transpose gives us

$$I_n = I_n^T = (AB)^T = B^T A^T.$$

Thus, B^T is the inverse of A^T if B is the inverse of A. Since $(A^T)^T = A$, we see that A is invertible if and only if A^T is invertible. From this point, the equivalence of all of these statements follows from previous theorems, mostly contained or mentioned in the previous section. □

You should definitely believe every one of the statements in this theorem is equivalent to all the others at this point. We've definitely proved all of these results independently. If you are the slightest bit suspicious[42] about any of these connections, you should find the theorem and definitions in sections past that prove it.[43]

42: I am!

43: Fine! I will!

Example 4.6.1 Let's check whether the following matrices are invertible. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 2 & 3 \end{bmatrix}$$

A is invertible for any of the eleven reasons on Theorem 4.6.4. B and C are not, again for any of the ten reasons.

Exploration 128 Determine whether each matrix below is invertible without performing any row operations. Note which part of the Invertible Matrix Theorem you use.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

Change of Basis Matrices

Our next topic is one we've seen in some examples, but we're finally ready to give it a proper discussion. We'll start, though, with an example.

Example 4.6.2 Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\},$$

which is a basis for $\mathbb{R}^3$. We can consider the coordinate map

$$\varphi_{\mathcal{B}} \colon \mathbb{R}^3 \to \mathbb{R}^3 \text{ defined as } \varphi_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}.$$

Then the matrix representation for $\varphi_{\mathcal{B}}$ with respect to the standard basis is

$$B = [[\vec{e}_1]_{\mathcal{B}} \ \dots \ [\vec{e}_2]_{\mathcal{B}}].$$

To find this, we need to compute all the coordinate vectors for the standard basis vectors with respect to $\mathcal{B}$. We can do this by augmenting the vectors of $\mathcal{B}$ with the standard basis vectors and row reducing. Does this sound familiar though? It should. Theorem 4.5.9 gave this as the algorithm for finding the inverse of a matrix. Let's see what this gives us.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & -2/3 \end{array}\right]$$

Then

$$C = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}.$$

It turns out that the matrix we were finding the inverse for is actually the matrix representation for $\varphi_{\mathcal{B}}^{-1}$, the linear transformation that converts a coordinate vector for $\mathcal{B}$ into a vector on the standard basis. Let's call our matrix representation of $\varphi_{\mathcal{B}}^{-1}$ here P since it's the one that's easier to find. Then we have

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = C.$$

With these two matrices, we can efficiently travel back and forth between the standard basis and our basis $\mathcal{B}$! We can even use these two matrices to convert any matrix representation for a linear transformation T from the standard basis to this new basis $\mathcal{B}$. We did this for a different basis back in Example 3.5.7, but now we can use matrix multiplication instead of those methods.

Example 4.6.3 Let's use P and P^{-1} from Example 4.6.2 above to compute the matrix representation for $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_2 \end{bmatrix}$$

with respect to the basis $\mathcal{B}$. First, we need the matrix with respect to the standard basis.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now, we should think about how to construct the desired matrix. When we think about this matrix as a function, it will have input of coordinate vectors for $\mathcal{B}$ and output of coordinate vectors for $\mathcal{B}$. Consider then the product $P^{-1}AP$.

- Since we read function composition right to left, we see that this will first apply P, which converts a coordinate vector for $\mathcal{B}$ to a vector in the standard basis.
- Then A will map this according to the linear transformation T.
- Finally, by applying P^{-1}, we convert the output of A to be a coordinate vector for $\mathcal{B}$.

This then will produce the matrix representation for T with respect to the basis $\mathcal{B}$.

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4/3 & -2/3 & -2/3 \\ 1/3 & 4/3 & 4/3 \\ 1/3 & -5/3 & -5/3 \end{bmatrix}. \end{aligned}$$

Since we have an alternative method for finding this matrix representation, we can check that this method produces the same matrix representation. By "we" there, we mean you. You should check this.[44]

44: Explore!

Exploration 129 Use the method from Example 3.5.7 to find the matrix representation of T above with respect to the basis $\mathcal{B}$.

The matrices P and P^{-1} were examples of *change of basis matrices*. Specifically, P^{-1} is the change of basis matrix from the standard basis to the basis $\mathcal{B}$ and P is the change of basis matrix from the basis $\mathcal{B}$ to the standard basis. We can do this, though, with any two bases.

Definition 4.6.2 *Let V be an n-dimensional vector space with bases*

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\} \text{ and } \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Define the isomorphism

$$\varphi_{\mathcal{B}\triangleright\mathcal{C}} : V \to V \text{ by } \varphi_{\mathcal{B}\triangleright\mathcal{C}}(\vec{c}_i) = \vec{b}_i \text{ for each } 1 \leq i \leq n.$$

*Then the **change of basis matrix from** $\mathcal{B}$ **to** $\mathcal{C}$ is the matrix for $\varphi_{\mathcal{B}\triangleright\mathcal{C}}$ with respect to the basis $\mathcal{C}$. In particular, it is the matrix $P_{\mathcal{B}\triangleright\mathcal{C}} \in \mathcal{M}_{n\times n}$ defined* *by*

$$P_{\mathcal{B}\triangleright\mathcal{C}} = \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}}\right].$$

Note that the inverse of a change of basis matrix is again a change of basis matrix. Perhaps that should be stated as a theorem.

Theorem 4.6.5 *For any two bases $\mathcal{B}$ and $\mathcal{C}$ of a vector space V, we have*

$$P^{-1}_{\mathcal{B}\triangleright\mathcal{C}} = P_{\mathcal{C}\triangleright\mathcal{B}}.$$

PROOF. This follows directly from the definition since we can quickly see $\varphi^{-1}_{\mathcal{B}\triangleright\mathcal{C}} = \varphi_{\mathcal{C}\triangleright\mathcal{B}}$. Thus, the matrix representations will also be inverses of one another. □

When we combine the definition of our change of basis matrix with our method from Section 4.3 for computing the coordinate vectors, we see there is a handy algorithm like Theorem 4.5.9 that we can use to compute these matrices.

Theorem 4.6.6 *Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be bases for $\mathbb{R}^n$. Then the matrix*

$$[\vec{c}_1\ \vec{c}_2\ \cdots\ \vec{c}_n | \vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_n] \text{ row reduces to } [I_n | P_{\mathcal{B}\triangleright\mathcal{C}}].$$

Similarly,

$$[\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_n | \vec{c}_1\ \vec{c}_2\ \cdots\ \vec{c}_n] \text{ row reduces to } [I_n | P_{\mathcal{C}\triangleright\mathcal{B}}].$$

In the case that one of the two bases involved is the standard basis for $\mathbb{R}^n$, we see that one direction will require no work and the other will only require us to compute the inverse of a matrix. This was what happened in Example 4.6.3. Let's see an example between two non-standard bases.

Example 4.6.4 Let's just do two different bases of $\mathbb{R}^2$. Define

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\} \text{ and } \mathcal{C} = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

We can see quickly that these are both bases of $\mathbb{R}^2$. Now, we can construct $P_{\mathcal{B}\triangleright\mathcal{C}}$ and $P_{\mathcal{C}\triangleright\mathcal{B}}$ using Theorem 4.6.6.

$$\left[\begin{array}{rr|rr} 2 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

and

$$\left[\begin{array}{rr|rr} 1 & -1 & 2 & -1 \\ 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 2 \end{array}\right].$$

This tells us

$$P_{\mathcal{B}\triangleright\mathcal{C}} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \text{ and } P_{\mathcal{C}\triangleright\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}.$$

Note that these two matrices are inverses of each other, as they should be!

Now that we've defined the change of basis matrix and talked about how to find it, let's see what it does for us. Hopefully, it changes a basis.

Theorem 4.6.7 *Let $\mathcal{B}$ and $\mathcal{C}$ be bases for the vector space V. For any $\vec{x} \in V$,*

$$P_{\mathcal{B} \triangleright \mathcal{C}} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}} .$$

PROOF. Let $\vec{x} \in V$ and suppose $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Then

$$\begin{aligned} P_{\mathcal{B} \triangleright \mathcal{C}} [\vec{x}]_{\mathcal{B}} &= \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}} \right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\ &= a_1 \left[\vec{b}_1\right]_{\mathcal{C}} + \cdots + a_n \left[\vec{b}_n\right]_{\mathcal{C}} \\ &= \left[a_1 \vec{b}_1 + \cdots a_n \vec{b}_n\right]_{\mathcal{C}} = [\vec{x}]_{\mathcal{C}} . \end{aligned}$$

□

Oh, good. It is named appropriately! Let's see an example of this theorem in action. Note that we restricted Theorem 4.6.6 for simplicity to bases of $\mathbb{R}^n$, but this algorithm works for any vector space once coordinate vectors are computed.

Example 4.6.5 Here, let's consider two bases of $\mathbb{P}_2$. Let

$$\mathcal{B} = \{1, 1+x, x+x^2\} \text{ and } \mathcal{C} = \{x, 1+x^2, 1\}.$$

Then under the coordinate mapping using the standard basis, $\{1, x, x^2\}$, these become

$$\overline{\mathcal{B}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \overline{\mathcal{C}} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We can then apply Theorem 4.6.6 to these bases of coordinate vectors to find $P_{\mathcal{B} \triangleright \mathcal{C}}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

Thus, we have

$$P_{\mathcal{B} \triangleright \mathcal{C}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

We can now check that this works as expected. Consider $\vec{p} = 2 + x + x^2$. Then from inspection we can see

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ and } [\vec{p}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To see for ourselves that $P_{\mathcal{B}\triangleright\mathcal{C}}$ converts coordinate vectors for $\mathcal{B}$ into ones for $\mathcal{C}$, we compute

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as expected.

Now that we know these matrices convert between coordinate vectors for different bases, we can expand them to matrix representations of a linear transformation.

Corollary 4.6.8 *Let vector space V have bases $\mathcal{B}$ and $\mathcal{C}$ and $T\colon V \to V$ have matrix representation B with respect to $\mathcal{B}$ and C with respect to $\mathcal{C}$. Define the matrix $P \in \mathcal{M}_{n\times n}$ by*

$$P = P_{\mathcal{B}\triangleright\mathcal{C}} = \left[\left[\vec{b}_1\right]_{\mathcal{C}} \cdots \left[\vec{b}_n\right]_{\mathcal{C}} \right].$$

Then we know

$$C = PBP^{-1} \qquad \textit{and} \qquad B = P^{-1}CP.$$

This was what we did in Example 4.6.3, but it's worth seeing again.

Example 4.6.6 Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \\ x_2 \end{bmatrix}.$$

Let's find the matrix for this linear transformation with respect to two different bases of $\mathbb{R}^3$. First, the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. We have

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the matrix for the linear transformation on the basis $\mathcal{E}$ is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let's write the matrix for T now with respect to the basis

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have

$$T(\vec{b}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad T(\vec{b}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad T(\vec{b}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

but these are vectors in coordinates relative to the standard basis $\mathcal{E}$. We can augment a matrix and row reduce to convert these to coordinate vectors for

$\mathcal{B}$.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Thus, the matrix for the linear transformation on the basis $\mathcal{B}$ is

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

For notational simplicity, let's denote

$$P = P_{\mathcal{B} \triangleright \mathcal{E}} = \left[\left[\vec{b}_1\right]_{\mathcal{E}} \left[\vec{b}_2\right]_{\mathcal{E}} \left[\vec{b}_3\right]_{\mathcal{E}} \right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

After verifying that

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

one can check that

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B. \end{aligned}$$

Exploration 130 Let

$$\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

This is a basis of $\mathbb{R}^3$. What is $P_{\mathcal{B}_0 \triangleright \mathcal{E}}$, where $\mathcal{E}$ is again the standard basis?

Now[45], use the matrix $P^{-1}_{\mathcal{B}_0 \triangleright \mathcal{E}}$ to find $[\vec{x}]_{\mathcal{B}_0}$ when $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

45: Hey, have you seen Ricky?

No.

It's not like Ricky to not just be hanging around.

[gasping] Hey guys. I've been all over this chapter. The proofs are all there.

You know they're all hyperlinked, right?

Section Highlights

- For an $m \times n$ matrix, A, the span of the rows of A forms a subspace of $\mathbb{R}^n$ called $\text{Row } A$. See Definition 4.6.1 and Theorem 4.6.1.
- For any $m \times n$ matrix, A, there is an orthogonal decomposition of $\mathbb{R}^n$ into $\text{Row } A \oplus \text{Ker } A$ and an orthogonal decomposition of $\mathbb{R}^m$ into $\text{Col } A \oplus \text{Ker } A^T$. See Corollary 4.6.3 and Figure 4.3.
- There are many conditions equivalent to a matrix being invertible. See Theorem 4.6.4.
- If $\mathcal{B}$ and $\mathcal{C}$ are both bases for a vector space V, then the change of basis matrix $P_{\mathcal{B} \triangleright C}$ converts coordinate vectors for $\mathcal{B}$ into coordinate vectors for $\mathcal{C}$. See Definition 4.6.2.
- If A is the matrix representation with respect to a basis $\mathcal{B}$ for a linear transformation, then the matrix representation with respect to the basis $\mathcal{C}$ is $P_{\mathcal{B} \triangleright C} A P_{\mathcal{B} \triangleright C}^{-1}$. See Corollary 4.6.8 and Example 4.6.6.

Exercises for Section 4.6

4.6.1. Find $\operatorname{Ker} A$, $\operatorname{Row} A$, $\operatorname{Col} A$, and $\operatorname{Ker} A^T$ when A is the matrix below.

(a) $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

4.6.2. Do each of these for the bases and vectors below.

- Find the change of basis matrix P that converts from the basis $\mathcal{B}$ to the standard basis.
- Find P^{-1}.
- Use P^{-1} to find $[\vec{x}]_{\mathcal{B}}$.

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}; \quad \vec{x} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}; \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}; \quad \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

4.6.3. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Let $\mathcal{C}$ be the standard basis for $\mathbb{R}^3$.

(a) Find the matrix $P_{\mathcal{B} \triangleright \mathcal{C}}$ and the matrix $P_{\mathcal{C} \triangleright \mathcal{B}}$.

(b) Use $P_{\mathcal{B} \triangleright \mathcal{C}}$ and $P_{\mathcal{C} \triangleright \mathcal{B}}$ to convert each of these matrices to the basis $\mathcal{B}$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

4.6.4. Let V be a vector space with bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$. If

$$\begin{array}{rcrcrcr} \vec{c}_1 & = & \vec{b}_1 & + & 2\vec{b}_2 & + & \vec{b}_3 \\ \vec{c}_2 & = & -\vec{b}_1 & - & \vec{b}_2 & + & \vec{b}_3 \\ \vec{c}_2 & = & & & \vec{b}_2 & + & \vec{b}_3 \end{array}$$

For $\vec{x} \in V$, find a matrix P such that $P\,[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$. Find P^{-1} and verify that $P^{-1}\,[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$.

4.7 More Fun with Least Squares

Here's a matrix equation:

$$A\vec{x} = \vec{b}.$$

Depending on A and $\vec{b}$, sometimes this equation has exactly one solution or an infinite number of solutions. That's very nice; everyone likes solutions. However, from Corollary 4.2.4 we know it's also possible that this matrix equation has no solutions. That must be very frustrating for everyone involved. Surely there's something one can do in this case! We can't just *pretend* that solutions exist; Corollary 4.2.4 is pretty clear about solutions existing or *not* existing, as is the binary nature of existence. However, if you find you're stuck in that latter case, what's the next best thing to an honest solution?[46] Perhaps, maybe, just possibly. . . an *approximate* one will do.

46: A dishonest one! Some would argue honesty is also binary in nature.

Let's assume $A \in \mathcal{M}_{m\times n}$, so $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$. Let's also assume some fool picked a rotten matrix A and a lonely vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solutions.

Definition 4.7.1 *A* ***least squares solution*** *for the matrix equation $A\vec{x} = \vec{b}$ is a vector $\hat{\vec{x}} \in \mathbb{R}^n$ such that for all $\vec{x} \in \mathbb{R}^n$,*

$$\|A\hat{\vec{x}} - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|.$$

The ***least squares error*** *of a least squares solution is $\|A\hat{\vec{x}} - \vec{b}\|$.*

These are the vectors in $\mathbb{R}^n$ whose images are *as close to* $\vec{b}$ as any other image vector. Good. In the absence of a solution to $A\vec{x} = \vec{b}$, these least-squares solutions are the closest you can get to a solution, and the least squares error quantifies how close. This inequality should look familiar; we've already encountered a situation like this. According to Theorem 2.6.1, if a vector $\vec{b}$ is not in a subspace W, then the vector in W closest to $\vec{b}$ is $\mathrm{proj}_W\left(\vec{b}\right)$. See Figure 4.4. Using $\mathrm{Col}\ A$ as the subspace W, we have the following corollary.

Corollary 4.7.1 *The least squares solutions for $A\vec{x} = \vec{b}$ are the solutions to the equation*

$$A\vec{x} = \mathrm{proj}_{\mathrm{Col}\ A}\left(\vec{b}\right).$$

PROOF. By Theorem 2.6.1,

$$\|\,\mathrm{proj}_{\mathrm{Col}\ A}\left(\vec{b}\right) - \vec{b}\| \leq \|\vec{y} - \vec{b}\|$$

for all $\vec{y} \in \mathrm{Col}\ A$. By definition of column space, for every vector $\vec{y} \in \mathrm{Col}\ A$, there is a set of vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$, and $\mathrm{proj}_{\mathrm{Col}\ A}\left(\vec{b}\right) \in \mathrm{Col}\ A$. The result follows from defining $\hat{\vec{x}}$ to be the set of vectors such that $A\hat{\vec{x}} = \mathrm{proj}_{\mathrm{Col}\ A}\left(\vec{b}\right)$. □

Corollary 4.7.1 suggests a strategy for finding the least squares solutions; one only need to solve the equation $A\vec{x} = \mathrm{proj}_{\mathrm{Col}\ A}\left(\vec{b}\right)$. Of course, this means first finding, $\mathrm{proj}_{\mathrm{Col}\ A}\left(\vec{b}\right)$, and to do this efficiently, you need an orthogonal basis for $\mathrm{Col}\ A$. Suddenly, this feels like a lot of work.

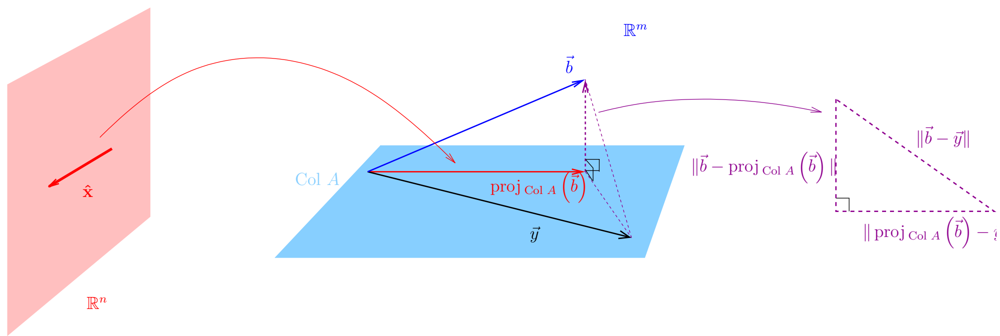

Figure 4.4: Here's a picture we've boosted from Section 2.6 and relabelled.

Example 4.7.1 Let's find the least squares solutions for the equation $A\vec{x}=\vec{b}$, where

$$A=\begin{bmatrix}1&2&3\\2&4&6\\0&0&1\end{bmatrix}\quad\text{and}\quad\vec{b}=\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

One should first check that $\vec{b}\notin\text{Col } A$. It's not? Good. Thank you for checking. If we're going to solve the equation $A\vec{x}=\text{proj}_{\text{Col } A}\left(\vec{b}\right)$, we'd better first calculate $\text{proj}_{\text{Col } A}\left(\vec{b}\right)$, so we need an orthogonal basis for Col A. After using Gram-Schmit, we have

$$\left\{\vec{w}_1=\begin{bmatrix}1\\2\\0\end{bmatrix},\vec{w}_2=\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

as an orthogonal basis for Col A. Then

$$\text{proj}_{\text{Col } A}\left(\vec{b}\right)=\frac{\vec{w}_1\cdot\vec{b}}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1+\frac{\vec{w}_2\cdot\vec{b}}{\vec{w}_2\cdot\vec{w}_2}\vec{w}_2=\frac{2}{5}\vec{w}_1+\vec{w}_2=\begin{bmatrix}2/5\\4/5\\1\end{bmatrix}.$$

We know $\text{proj}_{\text{Col } A}\left(\vec{b}\right)\in\text{Col } A$, so the matrix equation $A\vec{x}=\text{proj}_{\text{Col } A}\left(\vec{b}\right)$ definitely has at least one solution. We can just row reduce the augmented matrix

$$\left[\begin{array}{ccc|c}1&2&3&2/5\\2&4&6&4/5\\0&0&1&1\end{array}\right]\to\left[\begin{array}{ccc|c}1&2&0&-13/5\\0&0&1&1\\0&0&0&0\end{array}\right]$$

Thus, the least squares solutions for $A\vec{x}=\vec{b}$ are the vectors

$$\vec{x}=x_2\begin{bmatrix}-2\\1\\0\end{bmatrix}+\begin{bmatrix}-13/5\\0\\1\end{bmatrix},$$

for any real number x_2. These least squares solutions have least squares error

$$\|A\hat{\vec{x}}-\vec{b}\|=\|\text{proj}_{\text{Col } A}\left(\vec{b}\right)-\vec{b}\|=\left\|\begin{bmatrix}2/5\\4/5\\0\end{bmatrix}\right\|=\frac{2\sqrt{5}}{5}.$$

That didn't feel like a great way to find least squares solutions. Perhaps you're beginning to despair because you thought this way was our last hope. No, there is another...

We're all tired of writing $\text{proj}_{\text{Col } A}\left(\vec{b}\right)$ over and over again, right? Right. Let's define

$$\hat{\vec{b}} = \text{proj}_{\text{Col } A}\left(\vec{b}\right).$$

Then our least squares solutions are the vectors $\hat{\vec{x}} \in \mathbb{R}^n$ such that

$$A\hat{\vec{x}} = \hat{\vec{b}},$$

which looks like a totally respectable matrix equation.[47] By Theorem 2.5.1 (the orthogonal decomposition theorem!), we know that $\vec{b} - \hat{\vec{b}}$ is orthogonal to $\text{Col } A$. That is,

$$\vec{b} - \hat{\vec{b}} \in (\text{Col } A)^{\perp} = (\text{Span}\,\{\vec{a}_1, \ldots, \vec{a}_n\})^{\perp},$$

where $a_1, \ldots a_n$ are the columns of A. See Figure 4.4. It follows that $\vec{b} - \hat{\vec{b}}$ is orthogonal to every column of A; that is, for any column $\vec{a}_i$ of A, we have

$$0 = \vec{a}_i \cdot (\vec{b} - \hat{\vec{b}}) = (\vec{a}_i)^T(\vec{b} - \hat{\vec{b}}).$$

Since we have $(\vec{a}_i)^T(\vec{b} - \hat{\vec{b}}) = 0$ for every row vector $(\vec{a}_i)^T$, it follows that

$$A^T(\vec{b} - \hat{\vec{b}}) = \vec{0},$$

or $A^T\hat{\vec{b}} = A^T\vec{b}$. Recall that $A\hat{\vec{x}} = \hat{\vec{b}}$, so we have

$$A^TA\hat{\vec{x}} = A^T\vec{b}$$

for any $\hat{\vec{x}} \in \mathbb{R}^n$. Wow. That seems like a very conventional equation that we got by exploiting orthogonality. I wonder what we should call it.

47: Especially with all the fancy hats!

Definition 4.7.2 *The* ***normal equation*** *for a matrix $A \in \mathcal{M}_{m\times n}$ and a vector $\vec{b} \in \mathbb{R}^m$ is*

$$A^TA\hat{\vec{x}} = A^T\vec{b}.$$

We just proved the following theorem.

Theorem 4.7.2 *For a matrix $A \in \mathcal{M}_{m\times n}$ and a vector $\vec{b} \in \mathbb{R}^m$, a vector $\hat{\vec{x}} \in \mathbb{R}^n$ is a least squares solution for $A\vec{x} = \vec{b}$ if and only if $\hat{\vec{x}}$ is a solution to the normal equation*

$$A^TA\hat{\vec{x}} = A^T\vec{b}.$$

Since Corollary 4.7.1 indicates that least squares solutions to $A\vec{x} = \vec{b}$ are solutions to $A\vec{x} = \text{proj}_{\text{Col } A}\left(\vec{b}\right)$, and $\text{proj}_{\text{Col } A}\left(\vec{b}\right) \in \text{Col } A$, we know that least squares solutions always exist. Thus, we have another fun corollary:

Corollary 4.7.3 *For a matrix $A \in \mathcal{M}_{m\times n}$, the normal equation $A^TA\hat{\vec{x}} = A^T\vec{b}$ always has at least one solution.*

Example 4.7.2 Let's find the least squares solutions (again!) for the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

By Theorem 4.7.2, we want to find the solutions to $A^T A\hat{\vec{x}} = A^T\vec{b}$. Note that

$$A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 46 \end{bmatrix} \text{ and}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}.$$

Thus, we need to solve the matrix equation

$$\begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 46 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix},$$

whose augmented matrix in reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -13/5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which definitely looks familiar. It follows that the least squares solutions to $A\vec{x} = \vec{b}$ are still the vectors

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -13/5 \\ 0 \\ 1 \end{bmatrix},$$

for any real number x_2.

Exploration 131 Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$

Find the least squares solution(s) and least squares error for $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$.

This particular method for finding least squares solutions is really handy when finding curves of best fit. Let's try!

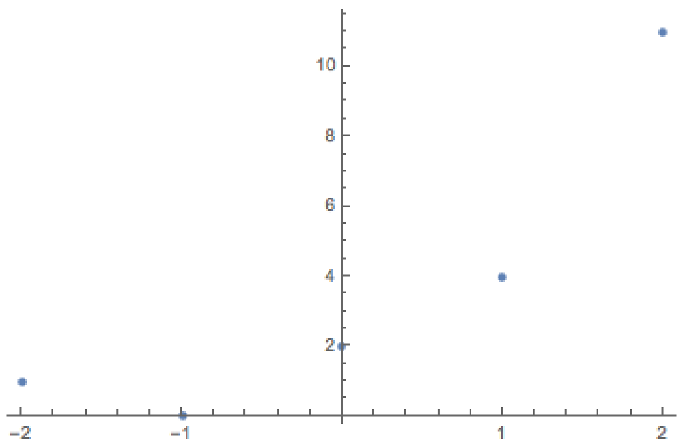

FIGURE 4.5. That old aggressively nonlinear data set again.

Example 4.7.3 Find (again!) the curve of best fit for the following data:

$$(-2,1),(-1,0),(0,2),(1,4) \text{ and } (2,11).$$

We checked before that these points are not colinear; see Figure 4.5.
Recall the example with the parabola of best fit? Sure you do. We need to find scalars a, b, and c such that $y = ax^2 + bx + c$ for all five of our points, but we know that no such a, b, and c exist; that is, there is no set of scalars a, b, and c such that

$$\vec{y} = a\vec{q} + b\vec{x} + c\vec{1}, \quad \text{where}$$

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 11 \end{bmatrix}, \quad \vec{q} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here are two other ways of putting it:

$$\vec{y} \notin \text{Span}\left\{\vec{q}, \vec{x}, \vec{1}\right\} = \text{Col}\,[\vec{q}, \vec{x}, \vec{1}],$$

or the matrix equation

$$[\vec{q}, \vec{x}, \vec{1}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{y}$$

has no solutions. Hey! No solutions? Let's find the least squares solutions! Let $A = [\vec{q}, \vec{x}, \vec{1}]$. Then

$$A^T A = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 34 & 0 & 10 \\ 0 & -10 & 0 \\ 10 & 0 & 5 \end{bmatrix}, \text{ and}$$

$$A^T \vec{y} = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 52 \\ 24 \\ 18 \end{bmatrix}.$$

Since we're looking for least squares solutions, we can solve the normal equation

$$\begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 52 \\ 24 \\ 18 \end{bmatrix},$$

whose augmented matrix in reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/7 \\ 0 & 1 & 0 & 12/5 \\ 0 & 0 & 1 & 46/35 \end{array}\right].$$

Hey! No Gram-Schmit required. Nice!
The quadratic equation $y = \frac{8}{7}x^2 + \frac{12}{5}x + \frac{46}{35}$ is the best quadratic least squares approximation for the given data. See Figure 4.6.

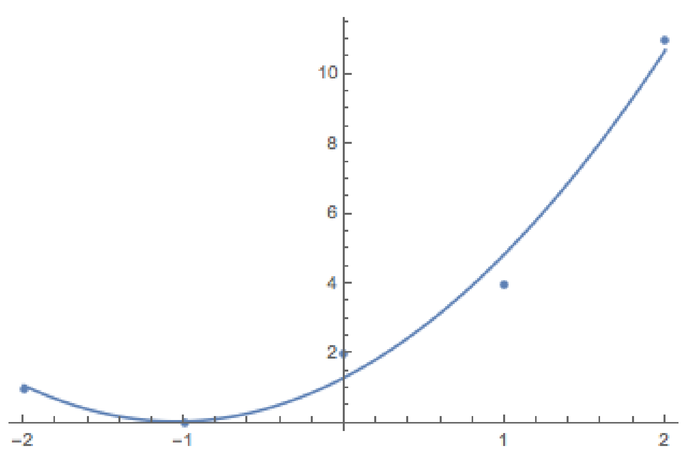

FIGURE 4.6. The last aggressively nonlinear data set with the parabola of best fit.

Section Highlights

- ► When a matrix equation $A\vec{x} = \vec{b}$ does not have a solution, an approximate solution can be found using the least squares solution. This is a solution to the matrix equation $A^T A\vec{x} = A^T\vec{b}$. See Definition 4.7.1 and Theorem 4.7.2.
- ► By converting data points into appropriate matrix equations, the technique of least squares can be used to find curves of best fit for data. See Example 4.7.3.

Exercises for Section 4.7

4.7.1. For any matrix $A \in \mathcal{M}_{m\times n}$, show that $A^T A$ is square and symmetric. That is, $A^T A \in \mathcal{M}_{n\times n}$ and $(A^T A)^T = A^T A$.

4.7.2. Suppose $A \in \mathcal{M}_{m\times n}$ and $\vec{b} \in \mathrm{Col}\ A$. Prove that the solutions to $A\vec{x} = \vec{b}$ are exactly the least squares solutions to $A\vec{x} = \vec{b}$.

4.7.3. Make a matrix $A \in \mathcal{M}_{4\times 3}$ with twelve reasonably nice integers. Find the least squares solutions and least squares error for $A\vec{x} = \vec{e}_1$.

4.7.4. Repeat the previous exercise.

4.7.5. Find the cubic curve of best fit for the following data:

$$(-2, 1), (-1, 6), (0, 2), (1, 4) \text{ and } (2, 11).$$

4.8 Another Graphics Application

Convolution and Edge Detection

Definition 4.8.1 *Let* $A = [a_{ij}]$, $B = [b_{ij}] \in \mathcal{M}_{m\times n}$. *The* **convolution** *of* A *and* B*, denoted* $A * B$*, is*

$$A * B = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}b_{ij}|.$$

Example 4.8.1 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix}.$$

Then

$$A*B = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}b_{ij}| = |-2a|+|3b|+|4c|+|-5d| = 2a+3b+4c+5d.$$

This probably seems like a weird thing to do with matrices. It is. Let's see just how weird. Fix a matrix $B \in \mathcal{M}_{m\times n}$ and define $T_B\colon \mathcal{M}_{m\times n} \to \mathbb{R}$ by $T_B(A) = A * B$. It turns out that T_B is *not* a linear transformation. Indeed, let

$$A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

One can readily check that

$$T_B(A+C) = 0 \neq 8 = T_B(A) + T_B(C).$$

Exploration 132 Is it true for any $k \in \mathbb{R}$ that $T_B(kA) = kT_B(A)$?

It's too bad for convolution that being "half linear" isn't a thing. It's worth noting that convolution is in part componentwise multiplication, and while this might seem like a reasonable way to define matrix multiplication, we see here that it definitely fails to preserve vector space properties in any reasonable way.

Definition 4.8.2 *Let*

$$G_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G_y = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

The **Sobel operator** *is the function* $S\colon \mathbb{M}_{3\times 3} \to \mathbb{R}$ *defined by*

$$S(A) = (A * G_x) + (A * G_y).$$

"Operator" is just another word for a function. It's most commonly used in conjunction with "linear;" the term "linear operator" usually refers to a linear transformation with a vector space of functions as its domain. In the case of Sobel operators, though, we know well that it just means "function" because the Sobel operator, S, is not a linear transformation.[48]

48: Exercise!

Example 4.8.2 Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\begin{aligned} S(A) &= (A * G_x) + (A * G_y) \\ &= |(a_{13} + 2a_{23} + a_{33}) - (a_{11} + 2a_{21} + a_{31})| \\ &\quad + |(a_{11} + 2a_{12} + a_{13}) - (a_{31} + 2a_{32} + a_{33})|. \end{aligned}$$

That's a pretty convenient way to calculate $S(A)$, but what does it mean? Why would anyone do something like this?

Let us experiment a bit with S. Perhaps we can find a use for it.

Example 4.8.3 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \text{and } C = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}.$$

One can check that

$$\begin{array}{lllllllll} G_x * A & = & 8 & G_x * B & = & 0 & G_x * C & = & 0 \\ G_y * A & = & 0 & G_y * B & = & 8 & G_y * C & = & 0 \end{array}$$

The last example illustrates the fact that $G_x * A$ quantifies how much the entries in a matrix A change *horizontally*. This is why $G_x * A$ is positive, and $G_y * B$ is zero; the entries of B don't change horizontally. Similarly, G_y quantifies how much the entries of a matrix A change *vertically*. This is something like a discrete version of a gradient of a function of two variables; we're quantifying the rate of change in the vertical and horizontal direction. Thus, the Sobel operator, S, quantifies the total change, both vertical and horizontal, in the entries of a 3×3 matrix.

Exploration 133 Find matrices A_1, A_2, and A_3 such that $S(A_i)$ is zero, small, and large, respectively.

The Sobel operator is a surprisingly effective detector of edges in images. Given a pixelated image (say of size 100×100), we can assign a value to each pixel to represent its color to create a matrix $A \in \mathcal{M}_{100\times 100}$. Then we apply the Sobel operator to every 3×3 section of A. Sufficiently large values for S would suggest the values in that section of A (that is, the colors in that section of the picture) were changing quickly either vertically or horizontally. This indicates the likely existence of an edge in the picture. Fun! Let's try it.

Example 4.8.4 Suppose we had a nice picture of a beautiful orange right triangle on a glorious brown plane. Using zeros for brown and ones for orange, we could construct a matrix $A \in \mathcal{M}_{12\times 12}$ to represent a badly pixilated version of such a picture. It might look something like

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&0&1&1&1&1&1&0\\ 0&0&0&0&0&1&1&1&1&1&1&0\\ 0&0&0&0&1&1&1&1&1&1&1&0\\ 0&0&0&1&1&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

See the triangle made of 1's? It's still pretty great; matrices make everything better. For $2 \le i, j \le 11$, let $A_{i,j}$ be the 3×3 submatrix of A centered at the entry in the ith row and jth column. For example, $A_{2,2}$ is the submatrix of entries in the shaded region in the upper left corner of the matrix, and $A_{6,6}$ is the submatrix of entries in the shaded region near the center of the matrix:

$$A_{2,2} = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix} \quad \text{and} \quad A_{6,6} = \begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&1&1 \end{bmatrix}.$$

Now let's construct a new matrix by applying the Sobel operator to each $A_{i,j}$:

$$\begin{bmatrix} S(A_{2,2}) & \cdots & S(A_{2,11})\\ \vdots & \ddots & \vdots\\ S(A_{11,2}) & \cdots & S(A_{11,11}) \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&2&6&4\\ 0&0&0&0&0&0&2&6&6&4\\ 0&0&0&0&0&2&6&6&2&4\\ 0&0&0&0&2&6&6&2&0&4\\ 0&0&0&2&6&6&2&0&0&4\\ 0&0&2&6&6&2&0&0&0&4\\ 0&2&6&6&2&0&0&0&0&4\\ 2&6&6&2&0&0&0&0&0&4\\ 6&6&2&0&0&0&0&0&0&4\\ 4&4&4&4&4&4&4&4&4&6 \end{bmatrix}$$

Returning to the original matrix A, we have colored white all the entries on the edge of the matrix (the Sobel operator was not defined for these entries), and we have also colored white all entries for which $S(A_{i,j}) < 4$. To be clear, we are just now seeing the entries that produce large (4 or greater) values with the Sobel operator, and we have nicely identified the boundary of the triangle.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is obviously a very simplified example for the purposes of being able to do the calculations by hand, but with minimal programing, one can implement this procedure very quickly with very high resolution images. See Figure 4.7.

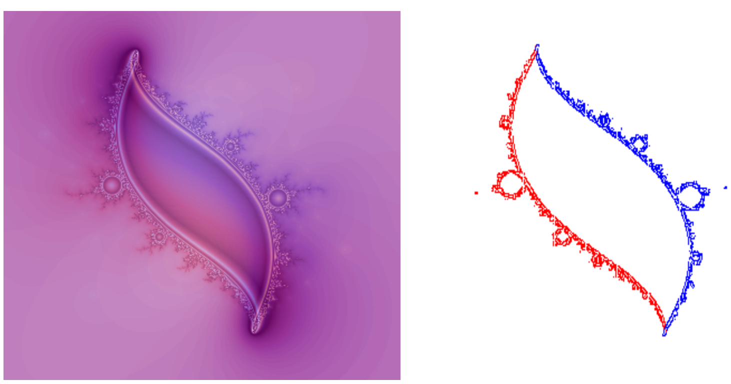

FIGURE 4.7. On the left is a fractal image. On the right is the boundary in the image, as identified using Sobel matrices.

Some optimized code and slightly more sophisticated linear algebra techniques (soon!) make applications like this very efficient. Again, this just scratches the surface of the power of linear algebra in computer graphics.