0 Functions on Sets

Functions are a ubiquitous tool in mathematics. They've been used by so many fields of thought and over such a vast swath of time, many forms of notation have evolved. Before we proceed, we should probably all get on the same page with functions. The following crash course in function theory is a deep dive both theoretically and notationally. Be sure to take the time to think critically about all of the definitions; this section is the foundation for all that comes later.

0.1 Sets

We will begin with some basic definitions and notation. It's imperative that the notation is understood since that is essentially the language we will be speaking.[1]

1: Well, and English. We will also speak English.

Hey, Ricky, what about telepathy?

Sorry, Bubbles. No telepathy... in this chapter.

Definition 0.1.1 *A* ***set*** *is an unordered collection of objects we call* ***elements****.*

Let's see a few familiar examples of sets.

- The set of integers, $\mathbb{Z}$, is the set of counting numbers, negative counting numbers, and 0. That is,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Note that we use brackets, $\{$ and $\}$, to enclose our set, and we list enough elements to see the pattern of what this set contains. We use the ellipsis, "$\dots$," at the beginning or end of our list (both in this case) to indicate that this pattern continues. Let us see a seemingly similar example.

- We don't have a standard name for this one, so we will use C to denote this set.

$$C = \{-2, -1, 0, 1, 2\}$$

While this looks similar to $\mathbb{Z}$ above, C is what's called a finite set. This set has only those 5 elements listed. We would read this as "C is the set that contains the elements -2, -1, 0, 1, and 2."

- The set of rational numbers, $\mathbb{Q}$, is the set of well-defined[2] ratios of integers. That is,

$$\mathbb{Q} = \left\{ \frac{p}{q} \colon p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

  We will also adopt the usual convention that two rational numbers are equal (that is, the *same* rational number) if they represent the same quantity. For example, $2/4 = 1/2$, so $2/4$ and $1/2$ are the same rational number.

Here, we again used brackets, $\{$ and $\}$, to bound our set description, but this time we were not able to list the elements. There are way too many! Instead, we used a general form with variables, $\frac{p}{q}$, with conditions on the variables, p and q. The colon is read as the phrase "such that" and separates the general form from the conditions. In some places (not this book), you will see a bar, $|$, used instead of the colon, but it means the same thing. The symbol, $\in$, is used to indicate when an element is in a set and is read as "is an element of" or "is in." The conditions in the above definition of $\mathbb{Q}$ says p and q are both elements of $\mathbb{Z}$, so they are integers.

- Let's see another example of this notation.

$$D = \left\{ \frac{n}{2} \colon n \in \mathbb{Z} \right\}$$

  This is read as "D is the set of all elements of the form $\frac{n}{2}$ such that n is an integer."

Note that since 2 is a nonzero integer, every element in this set, D, is also a rational number, so every element of D is in $\mathbb{Q}$.

Definition 0.1.2 *A* ***subset*** *of a set A is a subcollection of the elements of A; that is, B is a subset of A, written as $B \subseteq A$, if and only if every element of B is an element of A.*

From our discussion before, D is a subset of $\mathbb{Q}$, and we would denote this as $D \subseteq \mathbb{Q}$. Sometimes, the notation $\subset$ is used instead when we know that there are elements of the larger set that are not in the subset. This would be appropriate here. For example, $\frac{1}{3} \in \mathbb{Q}$ but $\frac{1}{3} \notin D$,[3] so we could alternatively say $D \subset \mathbb{Q}$.

Definition 0.1.3 *A* ***is equal as a set to*** *B, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.*

2: Ironically, "well-defined" is not defined well here. We just mean you can't divide by zero in your ratio.

3: It is common that a slash through a symbol means "not that symbol's meaning."

For example, ☎ clearly means "not a phone."

Example 0.1.1 ▶ It may seem obvious that $C = \{-2, -1, 0, 1, 2\}$ is not equal to $\mathbb{Z}$ just from their clearly different definitions. However, there are many (more subtle) situations in which this would not be quite so clear; such situations require a concrete definition. Let's practice using Definition 0.1.3 by verifying that C is not equal to Z. The intuition is that we need to show there is an element that is not common to both sets. Formally, note that $3 \in \mathbb{Z}$, but $3 \notin C$. Thus, $\mathbb{Z} \nsubseteq C$ and $C \neq \mathbb{Z}$.

▶ Consider these two sets

$$A = \{2n + 1 \colon n \in \mathbb{Z}\} \qquad B = \{2m + 3 \colon m \in \mathbb{Z}\}.$$

After writing out several elements for each set, one might begin to believe that both A and B are the set of odd integers and, therefore, equal. Let's use the definition of set equality to show that $A = B$. First, let $a \in A$. Then there is some $n \in \mathbb{Z}$ such that

$$\begin{aligned} a &= 2n + 1 \\ &= 2n - 2 + 2 + 1 \\ &= 2(n - 1) + 3. \end{aligned}$$

Since $n - 1 \in \mathbb{Z}$ for any $n \in \mathbb{Z}$, we know $a \in B$. This shows us that $A \subseteq B$. Now, we just need to start with $b \in B$ and argue that $b \in A$. This feels like a good place for an exploration!

Exploration 1 In the example above, we defined sets A and B and that $A \subseteq B$. Show that $B \subseteq A$ to complete the argument that these two sets are actually equal.

Exploration 2 Use Definition 0.1.3 to show $\mathbb{Z} \neq \mathbb{Q}$.

Definition 0.1.4 *Let A and B be sets and $B \subseteq A$. The **set difference of** A **and** B, denoted $A \backslash B$, is the set of elements in A and not in B. Specifically,*

$$A \backslash B = \{a \colon a \in A \text{ and } a \notin B\}.$$

Example 0.1.2 Let $A = \{n \in \mathbb{Z} \colon n < 0\}$, so A is the set of negative integers. Then $\mathbb{Z} \backslash A$ is the set of all integers except the ones in A; that is, the set of integers that *aren't* negative. Using set notation, we have

$$\mathbb{Z} \backslash A = \{n \in \mathbb{Z} \colon n \geq 0\}.$$

Definition 0.1.5 *Let A and B be sets. The* ***Cartesian product of*** *A* ***and*** *B, denoted $A \times B$, is the set of ordered pairs with the first component from A and the second from B. Specifically,*

$$A \times B = \{(a, b) \colon a \in A \text{ and } b \in B\}.$$

Example 0.1.3 Let's see an example of this. Let $E = \{1, 2\}$ and $F = \{a, b\}$. Then we have

$$E \times F = \{(1, a), (1, b), (2, a), (2, b)\}.$$

In this case, both E and F are finite sets, so $E \times F$ is also finite. Let's replace F here with a set that is not finite, say

$$G = \{1, 4, 7, 10, 13 \dots\}.$$

Then

$$E \times G = \{(1, 1), (1, 4), (1, 7), (1, 10), \dots (2, 1), (2, 4), (2, 7), (2, 10), \dots\}.$$

There is a very familiar example of a Cartesian product that we have not discussed. We call it $\mathbb{R}^2$, and it should have played a large role in many of the courses leading up to this one. First, let $\mathbb{R}$ denote the set of real numbers.[4] Then we can let both sets in the Cartesian product be $\mathbb{R}$ and form $\mathbb{R} \times \mathbb{R}$. In set notation, this is

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \colon x, y \in \mathbb{R}\}.$$

It will be convenient to refer to this as $\mathbb{R}^2$, mainly so we don't have to write as much!

Let's consider $E \times G$ above again for just a bit. Note that $E \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}$. It follows that $E \times G \subseteq \mathbb{R}^2$. We have a special name for this too!

Definition 0.1.6 *A* ***relation from*** *A* ***to*** *B, r, is a subset of $A \times B$; that is, $r \subseteq A \times B$.*

Let's take a step back now and explore why a subset might be termed a "relation." Suppose we were to ask a set of people what their favorite colors were. For privacy's sake, we will just use each person's first initial in our set of people:

$$P = \{I, S, O, W, R\}.$$

Our set of colors will be

$$C = \{\text{Red, Orange, Yellow, Blue, Green, Purple, Pink}\}$$

Now, we can use each person's response to form a subset of $P \times C$. Then

$$\begin{aligned} r &= \{(I, \text{Red}), (S, \text{Pink}), (O, \text{Red}), (W, \text{Green}), \\ &\quad (R, \text{Yellow}), (R, \text{Blue}), (R, \text{Green}), (R, \text{Purple})\}, \end{aligned}$$

and r is a relation. This word choice seems to fit this scenario very well. We are able to relate people and colors in this natural way and record it mathematically![5]

This idea of relating elements to each other probably seems pretty vague. That's actually a feature of this particular definition. It gives us flexibility so that we can define a relation by whatever goofy condition we want.

4: We will talk about this set quite a bit, so definitely note that notation.

I see what you did there. Very cute.

5: Note that in this example, R really only claimed the first three colors as his favorite and Purple as a secret favorite. There was no easy way to reflect this mathematically, though.

Example 0.1.4 Here are a couple of sets and a relation on them.

$$\begin{aligned} A &= \{1,2,3,4,5,6,7\} \\ B &= \{1,2,3,4,5,6,7\} \\ r &= \{(1,3),(2,4),(3,5),(4,6),(5,7)\} \end{aligned}$$

Let's describe this relation in words. Wait. Before reading the answer, do you see the pattern? The relation r relates elements in A to elements in B that are two greater.

Exploration 3 Now it's your turn! Let $A = \{1,3,5\}$ and $B = \{2,4,6,8\}$. Define a relation between the sets A and B that uses every element of A but not every element of B.

Example 0.1.5 Great! Now let's consider infinite sets. An example of a relation from $\mathbb{Z}$ to $\mathbb{Q}$ would be:

$$r = \left\{ \left(z, \frac{z}{2} \right) : z \in \mathbb{Z} \right\}.$$

Let's list out a few of the elements in this relation to see what it looks like.

$$(-2,-1), \left(-1, -\frac{1}{2}\right), (0,0), \left(1, \frac{1}{2}\right), (2,1), \left(3, \frac{3}{2}\right)$$

Note that the only elements of $\mathbb{Q}$ appearing here are actually elements in the subset D from earlier! Not every element in $\mathbb{Q}$ is paired with an element of $\mathbb{Z}$ here. Also note that each element of $\mathbb{Z}$ is paired with a distinct element of $\mathbb{Q}$. These are properties of relations that will be important for us soon.

Exploration 4 Now, define a relation between $\mathbb{Z}$ and $\mathbb{Q}$ yourself. Use set notation like above but also list a few elements of your relation.

Exercises for Section 0.1

0.1.1. All of the following statements are false. Give specific examples illustrating why they are false.

(a) $\mathbb{Q} \subset \mathbb{Z}$

(b) $\mathbb{R} \subset \mathbb{Q}$

(c) If $A \subseteq B$, then it's not possible for $A = B$.

0.1.2. Determine whether $x \in A$.

(a) $x = \dfrac{6}{5}$ and $A = \left\{ \dfrac{a+1}{a} \colon a \in \mathbb{Z} \right\}$

(b) $x = \dfrac{5}{3}$ and $A = \left\{ \dfrac{a+1}{a} \colon a \in \mathbb{Z} \right\}$

(c) $x = \dfrac{1}{2}$ and $A = \left\{ \dfrac{a}{4} \colon a \in \mathbb{Z} \right\}$

(d) $x = \dfrac{1}{6}$ and $A = \left\{ \dfrac{a}{2} + \dfrac{b}{3} \colon a, b \in \mathbb{Z} \right\}$

(e) $x = 3$ and $A = \{2a+1 \colon a \in \mathbb{Z}\}$

(f) $x = 4$ and $A = \{2a+1 \colon a \in \mathbb{Z}\}$

(g) $x = 4$ and $A = \{2a+1 \colon a \in \mathbb{R}\}$

0.1.3. Consider the sets below:

$$\begin{aligned} A =& \{a + b \colon a, b \in \mathbb{R}\} \text{ and} \\ B =& \{c - d \colon c, d \in \mathbb{R}\}. \end{aligned}$$

(a) Show $A \subseteq B$ by writing any sum of two real numbers as a difference.

(b) Show $B \subseteq A$ by writing any difference of two real numbers as a sum.

Congratulations! You've just proven that $A = B$!

0.1.4. Are these sets equal? Justify using the definition of set equality.

(a) $A = \{3n+1 \colon n \in \mathbb{Z}\}$ and $B = \{3m+4 \colon m \in \mathbb{Z}\}$

(b) $A = \{2n+1 \colon n \in \mathbb{Z}\}$ and $B = \{2m+2 \colon m \in \mathbb{Z}\}$

(c) $A = \left\{ \dfrac{a}{2} \colon a \in \mathbb{Z} \right\}$ and $\mathbb{Q}$

(d) $A = \left\{ \dfrac{a}{2} \colon a \in \mathbb{Q} \right\}$ and $\mathbb{Q}$

(e) $A = \left\{ \dfrac{a}{b} \colon a, b \in \mathbb{Q} \right\}$ and $\mathbb{Q}$

(f) $A = \{3x+1 \colon x \in \mathbb{R}\}$ and $\mathbb{R}$

(g) $A = \left\{ x^2 \colon x \in \mathbb{R} \right\}$ and $\mathbb{R}$

(h) $A = \left\{ x^2 \colon x \in \mathbb{R} \right\}$ and $B = \{x \colon x \in \mathbb{R}, x \geq 0\}$

0.1.5. Define a relation between the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y, z\}$.

0.1.6. Find all relations between the sets $C = \{1, 2\}$ and $D = \{a, b\}$.

0.2 Functions

We have all the pieces now for a nice, formal definition of functions. Let's see if we can make sense of this and relate it (pun intended) to our intuition about what a function is.

Definition 0.2.1 *Let A and B be sets. A* ***function from*** *A* ***to*** *B, often written $f\colon A \to B$, is a relation f from A to B such that*

$$\text{if } (a, b_1) \in f \text{ and } (a, b_2) \in f, \text{ then } b_1 = b_2.$$

The element $(a, b) \in f$ is often written $f(a) = b$.

Before we barrel[6] forward, let's see an example of this in action.

6: One of the two authors rejects the use of "barrel" as a verb.

Example 0.2.1 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Then the relation f given by

$$f = \{(1, 2), (3, 4), (5, 6), (7, 8), (9, 10)\}$$

is a function. We could also describe this function by

$$f(1) = 2,\ f(3) = 4,\ f(5) = 6,\ f(7) = 8,\ \text{and } f(9) = 10.$$

Example 0.2.2 Let's see a relation that is not a function with this definition. Again, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Define the relation r by

$$r = \{(1, 2), (1, 4), (3, 6), (5, 8), (7, 10)\}.$$

This is *not* a function because $(1, 2)$ and $(1, 4)$ are both in r, but $2 \neq 4$.

The notational convention at the end of Definition 0.2.1 (we write $(a, b) \in f$ as $f(a) = b$) allows us to rewrite the condition

"if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$" as
"if $f(a) = b_1$ and $f(a) = b_2$, then $b_1 = b_2$"

Furthermore, since $(a, b) \in f$ is taken to mean the same thing as $f(a) = b$, it is common to see a function defined by an algebraic formula. For example, if we were to define a function $f\colon A \to B$, where $A = B = \mathbb{Z}$, by the relation

$$f = \{(a, a + 3)\colon a \in A\},$$

we see that for any element $a \in A$, we have $(a, a + 3) \in f$, where $a + 3 \in B$. Thus, we could just as well define this relation by

$$f(a) = a + 3.$$

That probably looks a *lot* more familiar,[7] but what *is* the equation $f(a) = a + 3$? Is it the function? No. The function's name is f. The equation $f(a) = a + 3$ is an equation that defines the relation f; it tells us in practice how f relates elements from A to elements from B. This may seem like a trivial point, but a great deal of confusion springs forth from this subtlety in terminology. Here is a concrete breakdown of the terminology:

7: Notice that in this notation, the function from Example 0.2.1 would be described using $f(a) = a + 1$.

- The name of the function is f;
- a is an element of the first set A;
- $f(a)$ is an element of the second set B; and

- $f(a) = a + 3$ is an equation that indicates how to compute the element of B related to a by f.

These four things are *not* interchangeable. Understanding the difference between them will be particularly important for some of the discussion later in this text.

It's also worth underscoring that a function need not be defined by a formula. Indeed, if either the domain or the codomain is not a set of numbers, then a formula becomes quite impractical.

Exploration 5 Let r_1 and r_2 be relations from the set A to the set B given below:

$$\begin{aligned} A &= \{2, \sigma, +\}, \\ B &= \{1, 2, 3, 4, 5, 6, 7\}, \\ r_1 &= \{(2,3), (\sigma,3), (+,3), (2,6), (\sigma,6), (+,6)\}, \text{ and} \\ r_2 &= \{(2,3), (\sigma,3), (+,3)\}. \end{aligned}$$

Which of r_1 and r_2 are functions?

Exploration 6 Let r be the relation from $\mathbb{Q}$ to $\mathbb{Q}$ given by

$$r = \{(x^2, x) \colon x \in \mathbb{Q}\}.$$

Is r a function? Justify your response.

Definition 0.2.2 *Let $f \colon A \to B$ be a function. The* ***domain of*** *f is the set*

$$\begin{aligned} \operatorname{dom}(f) &= \{a \in A \colon \textit{there exists } b \in B \textit{ such that } (a,b) \in f\} \\ &= \{a \in A \colon \textit{there exists } b \in B \textit{ such that } f(a) = b\} \end{aligned}$$

Definition 0.2.3 *Let $f \colon A \to B$ be a function. The* ***range of*** *f is the set*

$$\begin{aligned} \operatorname{ran}(f) &= \{b \in B \colon \textit{there exists } a \in A \textit{ such that } (a,b) \in f\} \\ &= \{b \in B \colon \textit{there exists } a \in A \textit{ such that } f(a) = b\} \end{aligned}$$

Definition 0.2.4 *Let $f \colon A \to B$ be a function. The* ***codomain of*** *f, written* $\operatorname{codom}(f)$*, is the set B.*

This definition clears up a common ambiguity. If $f \colon A \to B$ is a function, must f relate *every* element of A to an element of B? Well, no. However, in practice, people often restrict the definition of their function to eliminate all the elements of A not related to an element of B by f. This is not always the case, though. Perhaps you have encountered functions like $f \colon \mathbb{Q} \to \mathbb{Q}$, defined by the equation $f(q) = 1/q$ where the the domain is incorrectly referred to as

$\mathbb{Q}$. Have you spotted the problem? While $0 \in \mathbb{Q}$, 0 cannot be related to some other rational number by the given equation. Thus, $\mathrm{dom}\,(f)$ in this example is actually $\mathbb{Q}\backslash\{0\}$, all rational numbers except 0.

Here's the takeaway from all of this. Given a function $f\colon A \to B$,

$$\begin{aligned} \mathrm{dom}\,(f) &\subseteq A, \text{ and} \\ \mathrm{ran}\,(f) &\subseteq B = \mathrm{codom}\,(f)\,. \end{aligned}$$

If you've never heard the word "codomain" before, *don't panic*. It's just another name for B, the second set in the relation f. Given an arbitrary function f, there's no reason to expect it to relate *all* the elements of A to *all* the elements of B, so we have these subsets $\mathrm{dom}\,(f)$ and $\mathrm{ran}\,(f)$ of A and B, respectively.

Lastly, it is common to think of elements of $\mathrm{dom}\,(f)$ as "inputs" and elements of $\mathrm{ran}\,(f)$ as "outputs" and to refer to the relating f does from A to B as "mapping." For example, given $f\colon \mathbb{Z} \to \mathbb{Z}$ by $f(z) = 9$, one could say the input -2 is mapped to the output 9. In fact, this very boring constant function maps all inputs to the output 9. If this makes you happy, then so be it. It does fit nicely with the notation $f\colon A \to B$.

Example 0.2.3 Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $\{(z, z^2)\colon z \in \mathbb{Z}\}$. This would be written as $f(z) = z^2$. Let's do the easy one first: $\mathrm{codom}\,(f) = \mathbb{Z}$. Now note that $\mathrm{dom}\,(f) = \mathbb{Z}$, and this is given explicitly in the set notation. However, using the notation $f(z) = z^2$, we must rely on the fact that there are no integers that cannot be squared in order to determine that $\mathrm{dom}\,(f) = \mathbb{Z}$. What about the range? Formally, we have

$$\mathrm{ran}\,(f) = \{z^2\colon z \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, 25, \ldots\},$$

that is, $\mathrm{ran}\,(f)$ is the set of all square integers. Lastly, note that our function f maps $\mathbb{Z}$ to $\mathbb{Z}$, so in the equation $f(z) = z^2$,

- f is the name of the function,
- $f(z) = z^2$ is an equation defining the relation between $\mathbb{Z}$ and $\mathbb{Z}$,
- z is an integer, and
- $f(z)$ is an integer (in $\mathrm{ran}\,(f)$); in particular, $f(z)$ is a squared integer.

Example 0.2.4 Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $f(z) = z^2/z^2$. You may be tempted to "simplify," whatever that means. Do not. Now, what's the domain of f? Are there integers that cannot be "plugged in" for z here? Yep! Our function, as it is defined, cannot handle 0. Thus, the domain is actually $\mathbb{Z}\backslash\{0\}$. Also, the range here is not all of $\mathbb{Z}$. What does f map 5 to for example? This function maps everything to 1. Thus, $\mathrm{ran}\,(f) = \{1\}$.

Exploration 7 Define the floor function $f\colon \mathbb{Q} \to \mathbb{Q}$ by $f(q) = \lfloor q \rfloor$, where $\lfloor q \rfloor$ is the greatest integer less than or equal to q. You can think of this function as the function that takes rational numbers and rounds them down to the nearest integer. Find the domain, range, and codomain for the function f. Note that these are all sets!

Operations on Sets

Now that we've talked about sets, Cartesian products, and functions, we can talk about something even more fun, binary operations on sets.

Definition 0.2.5 *Let A be a set. A* ***binary operation*** *on a set A is a function $f\colon A \times A \to A$, where the domain is $A \times A$.*

Some familiar examples are addition, subtraction, and multiplication.[8] If we write these in our standard function notation, we would have

$$\begin{aligned} +&\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \\ -&\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \text{ and} \\ \times&\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}. \end{aligned}$$

8: Yes, these stars of elementary school mathematics are about to get the full, formal treatment.

However, it is much more convenient and common to write $+(a, b)$ as $a + b$, and so on for the other symbols.[9]

9: Note that division is not listed here. That's because the old "can't divide by 0" rule makes this what's called a "partial binary operation."

An important question about operations is whether they are valid operations on a specific set. For example, let $A = \{x \in \mathbb{R} : x < 0\}$; that is, A is the set of negative real numbers. Then the operation of multiplication is not well-defined for this set. The product of two negative real numbers is a positive real number, which is not an element of A. This is not how we like our operations to behave; we would say in this case that A is **not closed under multiplication**.

Definition 0.2.6 *If $a * b \in A$ for any $a, b \in A$, we say the set A is* ***closed under the operation*** $*$.

Example 0.2.5 Let's consider the set $P = \{x \in \mathbb{R} : x \geq 0\}$. That is, P is the set of nonnegative real numbers. Since P is a subset of $\mathbb{R}$, all of our operations above make sense, but they may not give us an output that is also in P. Thus, we have to ask whether our set P is closed under these standard operations.

Addition:
To check this, we need to start with two elements of our set, so suppose $a, b \in P$. That means a and b are nonnegative real numbers. Must $a + b$ still be a nonnegative real number? Yes, yes it must. We know the sum of two real numbers is again a real number, and the sum of two positive numbers will again be positive. Also, 0 added to a positive number will just still be that positive number.

Subtraction:
For this one, the operation is not closed. We can see this by giving an example where the difference of two positive real numbers is no longer positive. First, note that $5, 10 \in P$. Then consider $5 - 10 = -5$. Since $-5 \notin P$, our set is not closed under this operation.

Exploration 8 Is our set P above closed under the operation of multiplication of real numbers?

Exploration 9 Can you think of a subset of $\mathbb{R}$ that is closed under addition but not under multiplication?

Exercises for Section 0.2

0.2.1. All of the following statements are false. Give specific examples illustrating why they are false.

(a) Every relation from a set A to a set B is a function.

(b) It's not possible for a function to have the same domain and codomain.

(c) It's not possible for a function to have the same codomain and range.

0.2.2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 8, 16\}$. Determine whether each of the following is a relation from A to B, a function from A to B, or neither.

(a) $r_1 = \{(1, 2), (2, 4), (3, 8), (4, 16), (5, 16)\}$

(b) $r_2 = \{(1, 16), (1, 8), (2, 16), (2, 8), (3, 16), (3, 8)\}$

(c) $r_3 = \{(1, 2), (2, 3), (3, 5), (4, 4)\}$

(d) $r_4 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$

0.2.3. Let $f\colon \mathbb{Q} \to \mathbb{Z}$ be defined by $f\left(\frac{a}{b}\right) = a$ for any $\frac{a}{b} \in \mathbb{Q}$. Is this a function?

0.2.4. Let $g\colon \mathbb{Q} \to \mathbb{Q}$ be defined by $g\left(\frac{a}{b}\right) = \frac{a+b}{b}$ for any $\frac{a}{b} \in \mathbb{Q}$. Is this a function?

0.2.5. Consider $A = \{1, 2, 3\}$ and $B = \{1, 3, 4, 5\}$. Is there a function $f\colon A \to B$ with $\operatorname{ran}(f) = B$? Give the function or explain why it does not exist.

0.2.6. For each function below, state the domain, codomain, and range of the function.

(a) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$

(b) $f\colon \mathbb{Z} \to \mathbb{Q}$ defined by $f(n) = \frac{n}{2}$ for all $n \in \mathbb{Z}$

(c) $f\colon \mathbb{Z} \to \mathbb{Q}$ defined by $f(n) = \frac{n+1}{3}$ for all $n \in \mathbb{Z}$

(d) $f\colon \mathbb{Q} \to \mathbb{Q}$ defined by $f(r) = \frac{r}{2}$ for all $r \in \mathbb{Q}$

(e) $f\colon \mathbb{Q} \to \mathbb{Q}$ defined by $f(r) = \frac{2}{r}$ for all $r \in \mathbb{Q}\backslash\{0\}$

(f) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x$ for all $x \in \mathbb{R}$

(g) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5$ for all $x \in \mathbb{R}$

(h) $f\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x) = (5, 5x)$ for all $x \in \mathbb{R}$

0.2.7. Determine whether the set A is closed under the given operation $\odot$.

(a) $A = \mathbb{Q}$ and $a \odot b = ab$ for any $a, b \in A$

(b) $A = \{n^2 : n \in \mathbb{Z}\}$ and $a \odot b = ab$ for any $a, b \in A$

(c) $A = \{n : n \in \mathbb{Z}, n \leq 0\}$ and $a \odot b = ab$ for any $a, b \in A$

(d) $A = \{n : n \in \mathbb{Z}, n \leq 0\}$ and $a \odot b = a + b$ for any $a, b \in A$

(e) $A = \{n : n \in \mathbb{Z}, n \leq 0\}$ and $a \odot b = a - b$ for any $a, b \in A$

(f) $A = \{x : x \in \mathbb{R}, x \geq 0\}$ and $a \odot b = a - b$ for any $a, b \in A$

(g) $A = \{x : x \in \mathbb{R}, x \geq 0\}$ and $a \odot b = ab$ for any $a, b \in A$

(h) $A = \{x : x \in \mathbb{R}, x > 0\}$ and $a \odot b = ab$ for any $a, b \in A$

(i) $A = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}$ and $a \odot b = ab$ for any $a, b \in A$

0.2.8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that $f(ab) = af(b)$ for any $a, b \in \mathbb{R}$.

(a) What must $f(0)$ be?

(b) There are only two options for $\operatorname{ran}(f)$. Can you name them?

0.2.9. Suppose $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has the related property that $g(ab, ac) = ag(b, c)$. To what must g map $(0, 0)$?

0.2.10. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(a, b) = a + b + ab$.

(a) Find an (a, b) such that $f(a, b) = 5$.

(b) Now, for any real number, x, find an appropriate (a, b) such that $f(a, b) = x$ for any real number x. This will show that $\operatorname{ran}(f) = \mathbb{R}$.

(c) Is the (a, b) from part (a) unique? If so, why? If not, find a different (a, b) such that $f(a, b) = 5$.

0.2.11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(a) = 2a + a^2$.

(a) Find an a so that $f(a) = -1$.

(b) Is there an a such that $f(a) = -2$?

(c) Suppose $f(a) = f(b)$. Must $a = b$?