5 Square Matrices and Invariant Subspaces

It is extremely common in applications for a linear transformation to have the same vector space for its domain and codomain. In such a case, the standard matrix for this linear transformation will be a square matrix. In this chapter, we focus our attention on this very case. While most of our attention will be paid to matrix transformations, keep in mind this can all be applied to any linear transformation from a vector space to itself once we find a matrix representation for the linear transformation.

5.1 Eigenvalues and How to Find Them

In general, when we multiply a matrix A by a vector $\vec{x}$, it's not easy to know what the result will be before doing the computation. Will $A\vec{x}$ have a different magnitude than $\vec{x}$? Probably. Will it have a different direction? Yeah, probably.[1] When can we expect $A\vec{x}$ to maintain some of the properties of $\vec{x}$? This is actually a pretty common question in mathematics; many mathematical questions boil down to some kind of *invariance* issue like this.

1: Recall that quantities with magnitude and direction were a way to think about vectors from back in Chapter 1. Magnitude correlates to length and direction can be represented with a unit vector.

Eigenvalues and Eigenvectors

As we so often do, let us begin with an example.

Example 5.1.1 Consider the matrix and vector below.

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Okay, can you guess what happens when we multiply $\vec{x}$ by the matrix B? Did you do the computation in your head to figure it out? That's cheating.

Let's compute $B\vec{x}$.

$$B\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

That's pretty cool. Multiplying by the matrix really just scales the vector by 3, so B preserves the direction of $\vec{x}$. Is B special, or is $\vec{x}$? What happens with other vectors? Let's check a few others.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Well, that's interesting. We found another one it scales by 3, one it scales by 1, and two that it seems to just change completely. There seems to be something special about some vectors relative to B. We should explore this more.

We state the following definition with the general term *scalar*, but for us, since our vector spaces are over $\mathbb{R}$, this means a real number.

Definition 5.1.1 *An* ***eigenvector*** *of a matrix $A \in \mathcal{M}_{n\times n}$ is a nonzero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an* ***eigenvalue*** *of A if there is a nontrivial solution $\vec{x} \in \mathbb{R}^n$ of $A\vec{x} = \lambda\vec{x}$, and we call such an $\vec{x}$ an* ***eigenvector corresponding to*** *λ.*

Think about that $A\vec{x} = \lambda\vec{x}$ equation for a second. This is saying that when you multiply $\vec{x}$ by the matrix A, it's the same as just rescaling $\vec{x}$ by λ. That's dynamite! Think of all the things A could do to $\vec{x}$, and yet, it just rescales $\vec{x}$.

Example 5.1.2 Let

$$A = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Is $\vec{x}$ an eigenvector of A? Behold,

$$A\vec{x} = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -2\vec{x}.$$

The matrix A rescales the vector $\vec{x}$ by -2, so yes, $\vec{x}$ is an eigenvector of A with eigenvalue $\lambda = -2$.

Example 5.1.3 Let

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}.$$

Is $\lambda = 2$ an eigenvalue of A? This is a slightly more difficult question. We need to know if there are *nontrivial* solutions to $A\vec{x} = 2\vec{x}$. We could just solve the associated system of equations. Alternatively, note that

$$\begin{aligned} A\vec{x} = \lambda\vec{x} &\qquad \text{if and only if} \\ A\vec{x} - \lambda\vec{x} = \vec{0} &\qquad \text{if and only if} \\ (A - \lambda I)\vec{x} = \vec{0}. & \end{aligned}$$

where I here is the 2×2 identity matrix. This is a more familiar problem. We're looking for the kernel of the new matrix $A-2I$. That is, $\vec{x} \in \mathrm{Ker}\,(A-2I)$ if and only if $A\vec{x} = 2\vec{x}$. Observe that

$$A - \lambda I = A - 2I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$\mathrm{Ker}\,(A-2I) = \mathrm{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}.$$

One can check that

$$\begin{aligned} A\begin{bmatrix} -2 \\ 1 \end{bmatrix} &= 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \\ A\begin{bmatrix} -4 \\ 2 \end{bmatrix} &= 2\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \text{ and} \\ A\begin{bmatrix} 200 \\ -100 \end{bmatrix} &= 2\begin{bmatrix} 200 \\ -100 \end{bmatrix}. \end{aligned}$$

In fact, for any scalar $k \in \mathbb{R}$ and any $\vec{x} \in \mathbb{R}^2$, we know that $A(k\vec{x}) = kA\vec{x}$ so

$$A\left(k\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = 2\left(k\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right).$$

That example illustrates that if you have an eigenvector corresponding to λ, then you definitely have more than one. Specifically, if $\mathrm{Ker}\,(A-\lambda I)$ contains a nonzero vector, then $\mathrm{Ker}\,(A-\lambda I)$ is a nontrivial subspace.

Definition 5.1.2 *The set of all solutions of*

$$(A - \lambda I)\vec{x} = \vec{0}$$

is a subspace of $\mathbb{R}^n$ *called the* ***eigenspace corresponding to*** λ *relative to the matrix* A*.*

Theorem 5.1.1 *For any* $n \times n$ *matrix* A*, the eigenspace corresponding to* λ *is a subspace of* $\mathbb{R}^n$*. Note that it is a nontrivial subspace if and only if* λ *is an eigenvalue for* A*.*

PROOF. This follows from the fact that the eigenspace corresponding to λ is $\mathrm{Ker}\,(A-\lambda I)$, which is a subspace. We call λ an eigenvalue of A exactly when there are nontrivial solutions to $A\vec{x} = \lambda\vec{x}$. Thus, this will be a subspace of at least dimension one (i.e. nontrivial) exactly when λ is an eigenvalue. $\square$

Example 5.1.4 Let

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}.$$

Let's find a basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 3$.

$$A - 3I = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the eigenspace corresponding to $\lambda = 3$ is

$$\text{Ker}\,(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\},$$

so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis.

Example 5.1.5 Let's try that again for the same matrix A, but suppose that $\lambda = 1$.

$$A - 1I = \begin{bmatrix} 3 & 0 & -1 \\ 3 & -1 & 3 \\ 2 & -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 0 & -2 & 14/3 \\ 0 & 0 & 5/3 \end{bmatrix}$$

This matrix has a pivot in every column, so its kernel is trivial. What this calculation shows us is that $\lambda = 1$ is not a valid eigenvalue for the matrix A since $A - 1I$ does not have nontrivial solutions.

Exploration 134 Without calculation, find one eigenvalue and two linearly independent eigenvectors of

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Hint: If $\text{Ker}\ A$ is nontrivial, then 0 is an eigenvalue with eigenspace $\text{Ker}\ A$.

We've seen now that eigenspaces are subspaces. Let $A \in \mathcal{M}_{n\times n}$, and suppose E_1 is the eigenspace corresponding to the eigenvalue λ_1 of A and E_2 is the eigenspace corresponding to the eigenvalue λ_2 of A, where $\lambda_1 \neq \lambda_2$. What could the intersection $E_1 \cap E_2$ look like? Well, suppose $\vec{x} \in E_1 \cap E_2$. That means $\vec{x} \in E_1$, so $A\vec{x} = \lambda_1\vec{x}$. Also, $\vec{x} \in E_2$, so $A\vec{x} = \lambda_2\vec{x}$. The only way both of these equations can be true is if $\vec{x} = \vec{0}$. Thus, the intersection of eigenspaces corresponding to distinct eigenvalues must be trivial. This idea motivates the following extremely useful[2] theorem.

2:  We'll get to the useful bit in the next section. Be patient.

Theorem 5.1.2 *If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly independent.*

PROOF. Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly dependent, so one of these vectors must be a linear combination of some of the others, which we may choose to be linearly independent. We can assume without loss of generality that $\vec{v}_1 = a_2\vec{v}_2 + \cdots + a_r\vec{v}_p$ for some scalars $a_2, \ldots, a_p$ (not all zero), where the vectors $\vec{v}_2, \ldots, \vec{v}_p$ are linearly independent. Then multiplying by A and using linearity of A and the fact that each $\vec{v}_i$ is an eigenvector corresponding to λ_i, we have

$$\begin{aligned} A\vec{v}_1 &= a_2 A\vec{v}_2 + \cdots + a_r A\vec{v}_p, \text{ so} \\ \lambda_1 \vec{v}_1 &= a_2 \lambda_2 \vec{v}_2 + \cdots + a_p \lambda_r \vec{v}_p. \end{aligned} \tag{5.1}$$

Multiplying $\vec{v}_1 = a_2\vec{v}_2 + \cdots + a_r\vec{v}_p$ by λ_1 and subtracting this from equation (5.1), we have

$$\vec{0} = a_2(\lambda_2 - \lambda_1)\vec{v}_2 + \cdots + a_p(\lambda_p - \lambda_1)\vec{v}_p.$$

Since the set $\{\vec{v}_2, \ldots, \vec{v}_p\}$ is linearly independent, we know that

$$a_2(\lambda_2 - \lambda_1) = \cdots = a_p(\lambda_p - \lambda_1) = 0.$$

Since the scalars $a_2, \ldots, a_p$ are not all zero, we must have $\lambda_1 = \lambda_i$ for some $2 \leq i \leq p$. This contradicts the fact that the eigenvalues are all distinct. Thus, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly independent. □

Corollary 5.1.3 *Suppose A is an $n \times n$ matrix with n distinct eigenvalues. Then the set of vectors formed by taking one eigenvector for each eigenvalue is a basis for $\mathbb{R}^n$. That is, there exists a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors for A.*

Eigenvalue Finding Algorithms

If you're working with small enough matrices, say in $\mathcal{M}_{2\times 2}$ or $\mathcal{M}_{3\times 3}$, there are some algebraic methods for finding both eigenvalues and eigenvectors. We're going to tell you about these methods…in the next section. Unless you're working with a matrix in $\mathcal{M}_{2\times 2}$ or maybe $\mathcal{M}_{3\times 3}$, you're almost certainly going to use technology to find eigenvalues and eigenvectors, so let's explore how that works a little bit. There are *many* algorithms for finding eigenvalues and eigenvectors that make various compromises in accuracy, complication, and speed. We'll just look at the most common and simple ones here, but know that the interested reader is free to fall down that rabbit hole if they wish, but we're not gonna push you.

The Power Method

This method will be used to find just a single eigenvalue and eigenvector, and as the name implies, we will use powers of matrices to help us in our computation. Before we start though, let's take note of a property of eigenvalues and eigenvectors related to matrix powers.[3]

3: I shall smite thee with my Matrix Powers!
No! Not the Matrix Powers! Ahh!

Theorem 5.1.4 *If $A \in \mathcal{M}_{n\times n}$ has eigenvalue λ with eigenvector $\vec{v}$, then*

$$A^k\vec{v} = \lambda^k\vec{v}$$

for any integer $k > 0$.

Rather than a formal proof, let's consider the case where $k = 2$. Then we have

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}.$$

The proof for the general statement works the same, just with bulkier notation. Now, using this theorem as motivation, let's see an example of a numerical approximation procedure for finding an eigenvector.

Example 5.1.6 We'll start with a 2×2 matrix and a randomly selected vector,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

That seems simple enough. Is $\vec{x}$ an eigenvector for A?

$$A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Well, it seems not. Certainly $A\vec{x}$ is not a scalar multiple of $\vec{x}$. Let's see what happens when we compute $A^k\vec{x}$ for large values of k even though we know $\vec{x}$ is not an eigenvector. Since A preserves direction for eigenvectors, we'll also include here the unit vector for each result, so we can compare how the direction is changing.

	Unit Vector
$A^2\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 11 \end{bmatrix}$	$\begin{bmatrix} 0.763386 \\ 0.645942 \end{bmatrix}$
$A^8\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^8 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 22,297 \\ 18,209 \end{bmatrix}$	$\begin{bmatrix} 0.774536 \\ 0.63253 \end{bmatrix}$
$A^{14}\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{14} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 37,567,993 \\ 30,674,171 \end{bmatrix}$	$\begin{bmatrix} 0.774596 \\ 0.632456 \end{bmatrix}$
$A^{20}\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 63,291,789,517 \\ 51,677,530,049 \end{bmatrix}$	$\begin{bmatrix} 0.774597 \\ 0.632456 \end{bmatrix}$

Well, that unit vector certainly seems to be converging to something. Could it be an eigenvector? How could we check? Let's see what A does to it.

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.774597 \\ 0.632456 \end{bmatrix} = \begin{bmatrix} 2.67197 \\ 2.18165 \end{bmatrix} \approx 3.4495 \begin{bmatrix} 0.774597 \\ 0.632456 \end{bmatrix}$$

So, not quite an eigenvector, but approximately one.

In the following theorem, we need to refer to the magnitude of an eigenvalue. If $\lambda \in \mathbb{R}$, then the *magnitude* is the absolute value, $|\lambda|$. This corresponds to how we referred to the magnitude of a vector in $\mathbb{R}$ back in Chapter 1.

Theorem 5.1.5 (Power Method) *Suppose $A \in \mathcal{M}_{n\times n}$ has n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ that can be arranged such that*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

Let λ_1 be the eigenvalue with the largest magnitude, then for any $\vec{x} \notin$ Span $\{\vec{v}_2, \ldots, \vec{v}_n\}$, $A^k\vec{x}$ approaches Span $\{\vec{v}_1\}$ *as k increases; that is, as k goes to infinity,*

$$\left\| A^k\vec{x} - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1\}} \left(A^k\vec{x}\right) \right\| \to 0.$$

What a cool name for a theorem. I'm impressed. This says that for almost any $\vec{x}$ you pick, $A^k\vec{x}$ will approach, as k increases, an eigenvector whose eigenvalue has the largest magnitude. This must be exactly what happened in Example 5.1.6. Let's prove it.

PROOF. We can order and number all of the eigenvalues by magnitude:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

Let $\vec{v}_1, \ldots, \vec{v}_n$ be the corresponding eigenvectors. Note that from Corollary 5.1.3 we know these eigenvectors form a basis for $\mathbb{R}^n$. Then for $\vec{x}$, we have $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ with $c_1 \neq 0$,[4] and

$$\begin{aligned} A^k\vec{x} &= A^k(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\ &= c_1A^k\vec{v}_1 + \cdots + c_nA^k\vec{v}_n \\ &= c_1\lambda_1^k\vec{v}_1 + \cdots + c_n\lambda_n^k\vec{v}_n. \end{aligned}$$

Factoring out λ_1^k, we have

$$\begin{aligned} A^k\vec{x} &= \lambda_1^k\left(c_1\vec{v}_1 + c_2\frac{\lambda_2^k}{\lambda_1^k}\vec{v}_2 + \cdots + c_n\frac{\lambda_n^k}{\lambda_1^k}\vec{v}_n\right) \\ &= \lambda_1^k\left(c_1\vec{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n\right). \end{aligned}$$

Since $|\lambda_i/\lambda_1| < 1$ for every $i = 2, \ldots, n$, we have that $(\lambda_i/\lambda_1)^k$ is very small. Thus, for very large k, $A^k\vec{x}$ is very close to $\lambda_1^kc_1\vec{v}_1 \in \operatorname{Span}\{\vec{v}_1\}$. More specifically,

$$\begin{aligned} \|A^k\vec{x} - \operatorname{proj}_{\operatorname{Span}\{\vec{v}_1\}}\left(A^k\vec{x}\right)\| &= \|A^k\vec{x} - \lambda^kc_1\vec{v}_1\| \\ &= \left\|c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n\right\|, \end{aligned}$$

which goes to zero as k goes to infinity. $\square$

More precise statements can be made and proved using limits. This is not calculus class, so we don't have to do that.[5] Nevertheless, Theorem 5.1.5 actually allows us to make an algorithm that rescales our vector at each step, and this rescaling allows it to converge to a specific vector as $k \to \infty$.

Corollary 5.1.6 *Suppose $A \in \mathcal{M}_{n\times n}$. Let λ_1 be the eigenvalue with the largest magnitude and unit eigenvector $\vec{v}_1$, and let $\{\vec{v}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. For any $\vec{x}_0 \notin \operatorname{Span}\left\{\vec{b}_2, \ldots, \vec{b}_n\right\}$, and any integer $k \geq 0$, define $\vec{x}_{k+1} = A\vec{x}_k/\|A\vec{x}_k\|$. Then*

$$\lim_{k\to\infty} \|\vec{x}_k - \vec{v}_1\| = 0.$$

Note that here the unit vectors are incorporated into the algorithm itself to make a nice convergence statement.[6] Once we have our eigenvector, $\vec{v}_1$, finding the eigenvalue is not so hard because $A\vec{v}_1 = \lambda_1\vec{v}_1$. Let's see another example.

4: We know $c_1 \neq 0$ because we assumed $\vec{x} \notin \operatorname{Span}\{\vec{v}_2, \ldots, \vec{v}_n\}$.

5: Note that Theorem 5.1.5 also works with complex eigenvalues by replacing absolute value with the complex number version of magnitude. However, we have simplified it here since we have so far only talked about vector spaces over $\mathbb{R}$.

So far?! Wait, are we going to talk about complex vector spaces?

Shhh! Not here.

6: It should also be noted here that convergence in norm like this only implies the convergence of the vectors here because our vector space is finite dimensional.

Wait. Things break when you need a basis with infinite vectors?

Yep. We're not going to talk about any of those things here, though.

Example 5.1.7 Now that we have the theorem, we can be a bit more efficient here. Let

$$B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

Now, if our theorem applies, we should see that applying a large enough power of B to some vector $\vec{x}$ will approximate an eigenvector. Let's try $k = 20$ and

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$B^{20}\vec{x} = \begin{bmatrix} 6,625,109 \\ -9,369,319 \end{bmatrix} \text{ with unit vector } \vec{u} = \begin{bmatrix} 0.57735 \\ -0.816497 \end{bmatrix}.$$

Now, we can check whether $\vec{u}$ is an approximate eigenvector.

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0.57735 \\ -0.816497 \end{bmatrix} = \begin{bmatrix} 1.39385 \\ -1.9712 \end{bmatrix} \approx 2.414 \begin{bmatrix} 0.57735 \\ -0.816497 \end{bmatrix}$$

Yay! We found an approximate eigenvalue and eigenvector.

Exploration 135 Use technology capable of computing powers of matrices to recreate the previous example with the matrix

$$C = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}.$$

While this method works almost all the time, you'll note that there were a few conditions on the theorem for when this exact technique applies and also on that initial vector choice. Also, be aware that we have worked examples of this with 2×2 matrices just for illustrative purposes. There are other techniques that are sometimes preferable for smaller matrices.

Perhaps you are sad that the Power Method only gives you one eigenvector and one eigenvalue. That is sad. It definitely makes me sad. Think of how left out all those other eigenvalues must feel. Fortunately, there are algorithms that find more eigenvalues. Let's see a different method.

The QR Method

For this method, we will need to *decompose* our matrix. This means we will write our matrix as the product of other matrices. We've already seen this in Chapter 4 when we discussed the change of basis matrices though we did not use this term, and we will discuss it quite a bit more in the upcoming sections. First, we need some handy definitions and theorems.

Definition 5.1.3 *The* ***main diagonal*** *of a square matrix $A \in \mathcal{M}_{n\times n}$ are the entries $a_{11}, a_{22}, \ldots a_{nn}$ starting at the upper left corner of the matrix and going diagonally to the lower right entry. A matrix is called* ***upper (lower) triangular*** *if all the entries below (above) the main diagonal are zero.*

Remember orthogonality from Chapter 2? What's better than orthogonality? Orthonormality!

Definition 5.1.4 *A matrix $A \in \mathcal{M}_{n\times n}$ is an* ***orthogonal matrix*** *if it has orthonormal columns.*

An "orthogonal" matrix has "ortho*normal*" columns? We concede this is confusing terminology but happily pass the buck to our mathematical ancestors. We'll gladly let this one slide because they also coined the terminology "spectral theory."[7] We'll talk about spectral theory soon. Anyway, orthogonal matrices, you will not be surprised, have very cool properties.

7: A term cool enough for superheroes to misuse so they sound smart.

Lemma 5.1.7 *If $A \in \mathcal{M}_{n\times n}$ is orthogonal, then $A^{-1} = A^T$.* [8]

8: Easiest inverse to calculate, ever!

What about I_n?

That's an orthogonal matrix, so it's inverse calculation is a subset of these calculations.

What calculation? It's its own inverse.

:...

PROOF. Let $A = [\vec{a}_1 \cdots \vec{a}_n]$. Using the definition of matrix multiplication from Theorem 4.4.5 gives that the a_{ij} entry of $A^T A$ is $\vec{a}_i \cdot \vec{a}_j$. The result follows immediately then from the fact that $\vec{a}_i \cdot \vec{a}_j = 0$ if and only if $i \neq j$, and $\vec{a}_i \cdot \vec{a}_j = 1$ if and only if $i = j$ since the columns are orthonormal. □

Theorem 5.1.8 (QR Decomposition) *If $A \in \mathcal{M}_{n\times n}$, then there is an orthogonal matrix $Q \in \mathcal{M}_{n\times n}$ and an upper triangular matrix $R \in \mathcal{M}_{n\times n}$ such that $A = QR$.*

PROOF. All we have to do is perform the Gram-Schmidt process to the columns of $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$.[9] Let $\vec{w}_1, \ldots, \vec{w}_n$ be the resulting vectors. Then normalize them (divide each of them by their own norm so the resulting vector has norm one); for $i = 1, \ldots, n$, define $\vec{u}_i = \vec{w}_i / \|\vec{w}_i\|$. It follows that $Q = [\vec{u}_1 \ \cdots \ \vec{u}_n] \in \mathcal{M}_{n\times n}$ is an orthogonal matrix. Now define the upper triangular matrix

9: Yeah! Gram-Schmidt those columns!

$$R = [\vec{r}_1 \ \cdots \ \vec{r}_n] = \begin{bmatrix} \vec{a}_1 \cdot \vec{u}_1 & \vec{a}_2 \cdot \vec{u}_1 & \cdots & \vec{a}_n \cdot \vec{u}_1 \\ 0 & \vec{a}_2 \cdot \vec{u}_2 & \cdots & \vec{a}_n \cdot \vec{u}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \vec{a}_n \cdot \vec{u}_n \end{bmatrix} \in \mathcal{M}_{n\times n}.$$

One can check that $A = QR$; behold that for any $i = 1, \ldots, n$, we have

$$\begin{aligned} Q\vec{r}_i &= [\vec{u}_1 \ \cdots \ \vec{u}_n] \begin{bmatrix} \vec{a}_i \cdot \vec{u}_1 \\ \vec{a}_i \cdot \vec{u}_2 \\ \vdots \\ \vec{a}_i \cdot \vec{u}_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= (\vec{a}_i \cdot \vec{u}_1)\vec{u}_1 + (\vec{a}_i \cdot \vec{u}_2)\vec{u}_2 + \cdots + (\vec{a}_i \cdot \vec{u}_i)\vec{u}_i = \vec{a}_i. \end{aligned}$$

□

That last equality is some sneaky business with Gram-Schmidt. Expect to work out the details as an exercise. Also, fun fact: there is a version of this theorem for rectangular matrices as well.[10] Note that the proof of this theorem, quite importantly, gives us a way *to find* the QR decomposition for a matrix, not just tells us that it exists. That's so helpful! Also, if you perhaps skipped over the proof, you should definitely go back and look at that. Let's see an example.

10: Non-square matrices are dead to me!

That's mean, Bubbles! Guess who's writing the proof for the general statement now?

. . . Um. . . the readers?

Oh, I was thinking *you*, but sure. That's fine, too.

Example 5.1.8 Let's find this decomposition for the matrix from Example 5.1.6,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}.$$

We'll refer to the columns here as $\vec{a}_1$ and $\vec{a}_2$, respectively. Now, the Gram-Schmidt process gives us

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and}$$

$$\vec{w}_2 = \vec{a}_2 - \operatorname{proj}_{\vec{w}_1}(\vec{a}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{5}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Normalizing these vectors gives us

$$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}.$$

Thus,

$$Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}.$$

Exploration 136 Verify that $A = QR$ with the matrices from the example above.

Now that we've learned about our decomposition, we can use it to find eigenvalues and eigenvectors. The following theorem gives us our algorithm.

Theorem 5.1.9 (QR Algorithm) *Suppose $A_0 \in \mathcal{M}_{n\times n}$ has n distinct eigenvalues such that*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0.$$

For any integer $k \geq 0$, let $Q_k R_k$ be the QR decomposition for A_k, and define

$$A_{k+1} = R_k Q_k.$$

Then the diagonal entries of A_k converge to the eigenvalues of A_0 as $k \to \infty$.

We're not going to prove this one; we'd need some machinery that's scattered throughout the next few sections. Thus, we will settle for seeing it in action. First, let's take note of a simplification. By using Lemma 5.1.7, the application of this theorem becomes a little easier in practice. Since $A_k = R_k Q_k$, we know $R_k = Q_k^{-1} A_k = Q_k^T A_k$ by Lemma 5.1.7. Then we have

$$A_{k+1} = Q_k^T A_k Q_k,$$

so we don't even need to compute R. That's neat! Now, it's time for that example.

Example 5.1.9 Here's a matrix,

$$A_0 = \begin{bmatrix} 5 & 6 & -6 \\ -7 & -12 & 15 \\ -4 & -8 & 11 \end{bmatrix},$$

with eigenvalues 3, 2, and -1. A quick[11] computation shows that, with entries rounded to the nearest thousandth,

$$\begin{aligned} A_1 &= \begin{bmatrix} 3.205 & 1.291 & 26.28 \\ 0.391 & 2.419 & 2.18 \\ -0.101 & -0.108 & -1.624 \end{bmatrix} \\ A_3 &= \begin{bmatrix} 2.92 & 0.604 & 26.172 \\ 0.215 & 2.141 & -4.046 \\ -0.011 & -0.007 & -1.061 \end{bmatrix} \\ A_{17} &= \begin{bmatrix} 3 & 0.334 & 24.288 \\ 0.001 & 2 & -10.582 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Note that in A_1 the diagonal entries are already getting near the eigenvalues. Did we cheat by starting with a matrix whose diagonal entries were pretty close to the eigenvalues already? Yes. Yes, we did. It's not uncommon to need hundreds of steps for this algorithm for reasonable precision.

This (or, technically, a more procedurally complicated, more efficient version[12] of this) is a super common way to find eigenvalues. However, you probably noticed that hypothesis about needing n distinct nonzero eigenvalues all with different magnitudes. Yeah. That's a strong one. It makes proving the convergence in this theorem possible, though. The good news is, *this algorithm works a lot of the time in general*, even without knowing anything about the eigenvalues.

Section Highlights

- Suppose A is an $n \times n$ matrix and $\vec{x} \in \mathbb{R}^n$. We say nonzero $\vec{x}$ is an eigenvector for A if $A\vec{x} = \lambda \vec{x}$ for some $\lambda \in \mathbb{R}$. In this situation, we call λ an eigenvalue for A. See Definition 5.1.1.
- A real number λ is an eigenvalue for A if and only if $\dim \operatorname{Ker}(A - \lambda I) \geq 1$. Moreover, any nonzero vector in $\operatorname{Ker}(A - \lambda I)$ is an eigenvector with eigenvalue λ. See Theorem 5.1.1.
- Any collection of eigenvectors corresponding to distinct eigenvalues form a linearly independent set. See Theorem 5.1.2.
- The power method and QR method are used to approximate eigenvalues of matrices. See Example 5.1.6 and Example 5.1.9.

11: Ha!

12: How can something more complicated be more efficient?

There are more steps in the algorithm, but it converges *much* faster.

Exercises for Section 5.1

5.1.1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$. Multiply each of the following vectors by A to determine whether or not they are eigenvectors for A. If they are, state the eigenvalue.

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, (g) $\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, (h) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

5.1.2. Ricky has affixed a picture of Bubbles onto the square in $\mathbb{R}^2$ whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. (He's done this at least once before.) He decides again that this is much too small, and he prefers that Bubbles faces the other direction. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that makes the picture of Bubbles twenty times bigger and reflects the image across the vertical axis. Find the eigenvalues for T. Justify. *Hint: No algebra required!*

5.1.3. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation that interchanges the x_1 axis with the x_3 axis, maps the x_2 axis to $\vec{0}$, and does nothing to the x_4 axis. Find all eigenvalues for T. Justify. *Hint: No algebra required!*

5.1.4. Determine whether or not 2 is an eigenvalue for each of the following matrices.

(a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 & \pi & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5.1.5. The following matrices all have eigenvalues 1 and 2. Find a basis for the eigenspace for each eigenvalue.

(a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

5.1.6. Let $B = \begin{bmatrix} -1 & 4 & -1 \\ 1 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix}$. The eigenvalues for B are -2, 2, and 1. Find an eigenvector corresponding to each eigenvalue.

5.1.7. Let $B = \begin{bmatrix} -1 & 4 & 0 \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix}$. The eigenvalues for B are -2, 3, and 0. Find an eigenvector corresponding to each eigenvalue.

5.1.8. The matrix

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ -3 & 7 & -6 & 3 \\ -3 & 5 & -4 & 3 \\ -3 & 4 & -6 & 6 \end{bmatrix}$$

has eigenvalues 1, 2, 3, and 4. According to Theorem 5.1.2, the associated eigenvectors should form a linearly independent set. Find an eigenvector for each eigenvalue, and verify that this set of four eigenvalues is linearly independent.

5.1.9. Explain why $A \in \mathcal{M}_{n\times n}$ can have at most n distinct eigenvalues.

5.1.10. Suppose $A^2 = I$. Show that the only possible eigenvalues are -1 and 1.

5.1.11. Determine whether or not the following matrices are orthogonal matrices.

(a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{bmatrix}$

(c) $C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 11/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{23} \\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$

5.1.12. Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$. Use the Power Method with $\vec{e}_1$ to approximate an eigenvector for the largest magnitude eigenvalue for A.

5.1.13. Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$. Find the QR decomposition for A.

5.1.14. Let $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$, and let $\{\vec{u}_1, \ldots, \vec{u}_n\}$ be the orthonormal set of vectors resulting from performing Gram-Schmidt on $\vec{a}_1, \ldots, \vec{a}_n$ and normalizing. Show that for $i = 1, \ldots, n$,

$$(\vec{a}_i \cdot \vec{u}_1)\vec{u}_1 + (\vec{a}_i \cdot \vec{u}_2)\vec{u}_2 + \cdots + (\vec{a}_i \cdot \vec{u}_i)\vec{u}_i = \vec{a}_i.$$

5.1.15. Adapt the statement and proof of Theorem 5.1.8 to hold for $A \in \mathcal{M}_{m\times n}$, where $m \neq n$.

5.2 Determinants and More Fun with Eigenvalues

It's been a while since we dealt with real-valued functions, so this will be a nice change.[13] The first part of this section is devoted to defining a function from $\mathcal{M}_{n\times n}$ to $\mathbb{R}$ that has some very convenient properties. It's called the *determinant*. Then, we will use this function as a tool to compute eigenvalues, as we promised in the previous section. Since our ultimate goal here will be our hunt for eigenvalues, we have omitted many of the proofs related to the determinant function.

13: Really? Somehow I'm skeptical.

Determinants

Here's a neat fact. The determinant is the *unique* function from $\mathcal{M}_{n\times n}$ to $\mathbb{R}$ that is n-linear, alternating, and maps the identity matrix I_n to 1. If we had a couple dozen extra pages we could figure out what all of that means and use it to define the determinant function. That would be really great. However, we're going to take a more…um…*equestrian* approach. Yeah, that's the word I wanted. This way has more horses. Tons more horses. [14]

14: How dare you betray us after all we've done for you! Horses?!

Oh? Did you think I meant *pedestrian*? Nope. Not at all. There is nothing the slightest bit pedestrian about determinants. Ultimately, we're just building a function here (pedestrian), but this function is neat and hairy (equestrian). However, rather than build the spike-wheeled chariot of alternating, multilinear functionals, we're going to build a more equestrian/pedestrian saddle of a function. Just know in your heart that if you wanted to turn in your determinant saddle for a determinant chariot, all you have to do is watch *Ben Hur* and look up "determinant" on Wikipedia.

Commence construction! Let $\det\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$, and let's make it so that for any $A \in \mathcal{M}_{2\times 2}$, we have $\det(A) \neq 0$ if and only if A is invertible.[15] From this, we can build a formula for $\det$. In Section 4.5, we saw that a matrix

15: Well, that's a handy property!

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}_{2\times 2}$$

has an inverse

$$\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} -a_{11} & a_{21} \\ a_{12} & -a_{22} \end{bmatrix}$$

if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Great! Then we define $\det\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ by

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

That was easy, like riding a bike (being pulled by a horse).[16] Here's a definition of a function $\det\colon \mathcal{M}_{n\times n} \to R$ that uses this $\det$ function we just defined on $\mathcal{M}_{2\times 2}$.

16: Strike that. The authors have it on good authority that you don't want to ride a bike being pulled by a horse. Ever.

Definition 5.2.1 *For $n \geq 2$, the* ***determinant*** *of a matrix $A = [a_{ij}] \in \mathcal{M}_{n\times n}$ is the sum*

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{n+1}a_{1n}\det A_{1n}\\ &= \sum_{j=1}^{n}(-1)^{j+1}a_{1j}\det A_{1j},\end{aligned}$$

where $A_{ij} \in \mathcal{M}_{(n-1)\times(n-1)}$ is the submatrix of A resulting from removing the ith row and jth column.

Defining the determinant this way allows us to maintain the property that $\det(A) \neq 0$ if and only if A is invertible; we'll prove that later in this section. It also turns out that this determinant function is unique (in a very specific sense). However, perhaps you are asking yourself the following question: What is this grossly complicated thing? That is a fair question. Hopefully an example will clarify.

Example 5.2.1 Let's calculate a determinant. Let $A \in \mathcal{M}_{3\times 3}$, so our formula from Definition 5.2.1 is

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}.$$

If

$$A = \begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix},$$

then

$$\begin{aligned} a_{11} = 1, &\quad A_{11} = \begin{bmatrix} 5 & 6\\ 8 & 9\end{bmatrix};\\ a_{12} = 2, &\quad A_{12} = \begin{bmatrix} 4 & 6\\ 7 & 9\end{bmatrix};\ \text{and}\\ a_{13} = 3, &\quad A_{13} = \begin{bmatrix} 4 & 5\\ 7 & 8\end{bmatrix}.\end{aligned}$$

It follows that

$$\begin{aligned}\det\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} &= \det\begin{bmatrix} 5 & 6\\ 8 & 9\end{bmatrix} - 2\det\begin{bmatrix} 4 & 6\\ 7 & 9\end{bmatrix} + 3\det\begin{bmatrix} 4 & 5\\ 7 & 8\end{bmatrix}\\ &= (-3) - 2(-6) + 3(-3) = 0\end{aligned}$$

That was anticlimactic. Neglect ye not the minus signs that alternate from submatrix to submatrix!

Exploration 137 Calculate the determinants of the matrices below.

$$A = \begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6\\ 0 & 8 & 9\end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3\\ 4 & 5 & 6\\ 0 & 8 & 9\end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0\\ 4 & 5 & 6\\ 400 & 8 & 9\end{bmatrix}$$

Did you appreciate the zeroes across the top row in that matrix C? Yes, we thought you would. Suppose those zeroes had been in the second row. It would have still simplified the computation, but not quite as much. It'd be great if we could just pick whichever row or column we wanted and use that to compute the determinant. We could always pick the one with the most zeroes. Well, guess what? We can do exactly that. Before stating this as a theorem, though, we need a definition.

Definition 5.2.2 *Let $A \in \mathcal{M}_{n\times n}$ and suppose A_{ij} is the submatrix formed by deleting the i^{th} row and j^{th} column from A. Then the number*

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

*is called the $(\mathbf{i}, \mathbf{j})-$**cofactor** of A.*

With this terminology and notation, we can write Definition 5.2.1 as

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

We call this a **cofactor expansion** of the determinant. Now, for that lovely theorem about picking your favorite row or column.

Theorem 5.2.1 *The determinant of a matrix $A \in \mathcal{M}_{n\times n}$ can be computed by a cofactor expansion across any row or down any column. In particular, the expansion across the i^{th} row is*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and the expansion down the j^{th} column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 5.2.2 Let's start with a matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Now, we can compute the determinant across any row or column. Let's pick row 2!

$$\begin{aligned} \det A =& -0\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} - 0\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \\ =& 2(0-3) = -6 \end{aligned}$$

Let's do it again! This time, we'll use column 3!

$$\begin{aligned} \det A =& 1\begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix} - 0\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ =& 1(0-6) = -6 \end{aligned}$$

Is 0 now your favorite number? Maybe it should be.

Now, let's talk about those negative signs. Did you catch in Example 5.2.2 that they alternated in the pattern $-, +, -$ when going across the second row but $+, -, +$ when going down the third column? Well, there's a nice way to remember where the negative signs go. Whatever size your matrix, you can build a checkerboard pattern to remind you how these work. Here are the checkerboards for 3×3 and 4×4 for reference.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Exploration 138 Compute the determinant by cofactor expansion along the row or column that involves the least amount of computation.

$$A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$$

That last theorem simplified our computations when we have a few zeroes. This next one will simplify it even more depending on where the zeroes are located.

Theorem 5.2.2 *The determinant of an $n \times n$ triangular matrix A is the product of the diagonal entries.*

PROOF. If $A \in \mathcal{M}_{n\times n}$ is upper triangular, the result follows from doing each cofactor expansion down the first column. If A is lower triangular, do each cofactor expansion down the last column. □

Example 5.2.3 Let's calculate some more determinants!

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & \pi^2 \\ 0 & 1 & 4 & 6 & 42 \\ 0 & 0 & 4 & 8 & e \\ 0 & 0 & 0 & 1 & 1,234 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By Theorem 5.2.2, $\det A = 4$.

This method was great, but it relied on having a very specific form for the matrix. Are we allowed to alter the form of the matrix? Well, yes and no. We have those handy row operations, but we'd need to know how those interact with the determinant. It seems we are ready to explore how this function behaves with some of the other things we've encountered.

Properties of Determinants

By carefully breaking down the definition of the determinant, we can say precisely how it is affected by each row operation.

Theorem 5.2.3 *(Row Operations and Determinants) Let A be a square matrix.*

(a) *If a multiple of one row of A is added to another to produce matrix B, then* $\det B = \det A$.

(b) *If two rows of A are interchanged to produce B, then* $\det B = -\det A$.

(c) *If one row of A is multiplied by k to produce B, then* $\det B = k \cdot \det A$.

Thus, we are allowed to modify the format of a matrix to help in calculating the determinant, but we'll need to carefully keep track of the row operations to do so.

Example 5.2.4 Let's use properties of determinants to simplify the computation of determinants.

$$A = \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$$

Using only the row operation of adding a multiple of one row to another, one can show that

$$A \to \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 10 \end{bmatrix} = B$$

By Theorem 5.2.3, $\det A = \det B$, and by using the cofactor expansion across the third row of B, we see that $\det B = 0$. Thus, $\det A = 0$.

Recall that elementary matrices allow us to convert row operations to matrix multiplication. Thus, we have the following corollary.

Corollary 5.2.4 *Suppose $A \in \mathcal{M}_{n\times n}$ and $E \in \mathcal{M}_{n\times n}$ is an elementary matrix. Then* $\det EA = \det E \det A$.

This proof here can be done by computing $\det E$ for each type of elementary matrix and comparing to Theorem 5.2.3, but we will leave this for the exercises. This corollary might have you questioning whether we can always break up the determinant across matrix multiplication. Well, yes, actually we can. Before we can prove that though, we need our promised connection between the determinant and invertibility.

Theorem 5.2.5 *A square matrix A is invertible if and only if* $\det A \neq 0$.

PROOF. This follows from Theorem 5.2.3. Let B be the row-echelon form for the matrix A. We can always obtain row-echelon form without rescaling the rows since we are not required to make the pivots 1s, and since adding a scalar multiple of a row does not change the determinant, we only need to keep track of the row swaps to compute the relationship between $\det A$ and $\det B$. Suppose r row swaps were needed. Then

$$\det A = (-1)^r \det B.$$

Now, A is invertible if and only if B has a pivot in every column, which means A is invertible if and only if B has no zeroes on its diagonal. Thus, the result follows from Theorem 5.2.2. □

Example 5.2.5 Use determinants to decide if the matrix is invertible.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

Since $\det A = -1 \neq 0$, A is invertible by Theorem 5.2.5.

Because of the Invertible Matrix Theorem, knowing whether or not a matrix is invertible can also tell us other interesting things.

Exploration 139 Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

Now, we have everything needed to expand Corollary 5.2.4 into a more general statement.

Theorem 5.2.6 *(Multiplicative Property) If $A, B \in \mathcal{M}_{n\times n}$, then*

$$\det AB = (\det A)(\det B)$$

PROOF. Let $A, B \in \mathcal{M}_{n\times n}$. Suppose first that A is not invertible. Then $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is not onto and $\operatorname{Imag} T_A \neq \mathbb{R}^n$. This means $T_{AB} = T_A \circ T_B$ cannot be onto since $\operatorname{Imag} T_{AB}$ must be a subset of $\operatorname{Imag} T_A$. Thus, AB is also not invertible. By Theorem 5.2.5, $\det A = 0$ and $\det AB = 0$. We can then conclude $\det AB = 0 = \det A \det B$.

Suppose now that A is invertible. Then A is row equivalent to the identity matrix and there is some sequence of row operations $\{r_1, \ldots, r_k\}$ such that $A = (E_{r_k} \cdots E_{r_1})I_n$ where E_{r_i} are each elementary matrices. We saw in Corollary 5.2.4 that the determinant breaks up across the product of an elementary matrix and a regular matrix, so we have

$$\begin{aligned} \det AB &= \det((E_{r_k} \cdots E_{r_1})I_n B) \\ &= (\det E_{r_k}) \cdots (\det E_{r_1})(\det I_n)(\det B) \\ &= (\det A)(\det B). \end{aligned}$$

□

Beware! There is no analogue for sums of matrices. That is, $\det(A+B) \neq \det A + \det B$ in general. We'll verify this in an exercise.

Exploration 140 Let's use Theorem 5.2.6 to find the relationship between $\det A$ and $\det A^{-1}$ when A is invertible.

Suppose we know $\det A = k$ for some $k \in \mathbb{R}$. If A is invertible, we know $k \neq 0$ from Theorem 5.2.5. Then there exists some A^{-1} such that $AA^{-1} = I_n$. From Theorem 5.2.2, what must $\det I_n$ be?

Finally, what must $\det A^{-1}$ be?

Exploration 141 Compute $\det A^5$ if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Let's see one last property before moving on to eigenvalues. From Theorem 5.2.1, we know $\det A$ can be computed using a cofactor expansion along any row or column. Thus, since the transpose operation just swaps rows and columns, the following theorem should not be surprising.

Theorem 5.2.7 *If $A \in \mathcal{M}_{n\times n}$, then $\det A^T = \det A$.*

Finding Eigenvalues with Algebra

You've probably noticed we haven't explicitly outlined a method for finding eigenvalues by hand. That must be very annoying. We should fix that. But first, more terminology!

Definition 5.2.3 *For $A \in \mathcal{M}_{n\times n}$, the degree n polynomial $\det(A-\lambda I)$ is called the* ***characteristic polynomial*** *for A.*

Yes, take the determinant. What does the determinant tell us? Well, if it's nonzero, our matrix is invertible. But do we want $A-\lambda I$ to be invertible? No! We want a nontrivial kernel for $A-\lambda I$. We very specifically want to figure out which λ give us a non-invertible matrix $A-\lambda I$. Thus, we need to know when this characteristic polynomial is zero. Let's bundle this up into a theorem.

Theorem 5.2.8 *For $A \in \mathcal{M}_{n\times n}$, λ is an eigenvalue of A if and only if it is a zero of the characteristic polynomial for A.*

Yay! We've turned this into the algebra problem of finding zeros for a polynomial. That's easy, right?[17]

17:  Right. . . This is actually an extremely deep problem. For a polynomial of degree two, we have the quadratic equation, and there are also formulas that exist for degree 3 and degree 4. One of the most groundbreaking results of the early 1800s is that no similar formula exists for polynomials of degrees 5 and higher. Mathematicians had searched for a solution for *hundreds* of years before it was proven none existed.

 So. . . maybe not easy?

 *Hundreds* of years? Is that supposed to be a long time?

 . . .

Example 5.2.6 Let's find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} -\lambda & 1 & 5 \\ 0 & 1-\lambda & 0 \\ 5 & 1 & -\lambda \end{bmatrix} \\ &= -\lambda[(1-\lambda)(-\lambda)] + 5[-5(1-\lambda)] \\ &= -\lambda^3 + \lambda^2 + 25\lambda - 25 = -(\lambda-1)(\lambda+5)(\lambda-5). \end{aligned}$$

Thus, we know that $\det(A - \lambda I) = 0$ when $\lambda = -5$, 1, and 5; thus, the eigenvalues of A are -5, 1, and 5.

Exploration 142 Find the eigenvalues of

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

using Theorem 5.2.8.

Great! This gives us a systematic approach to finding eigenvalues by hand.[18] But sometimes, our task is even easier. For example, if it is quickly apparent that the matrix is itself not invertible, then 0 is one of its eigenvalues. Also, if the matrix is triangular, either upper or lower triangular, we have the following result.

18:  Hem, hem. When $n \leq 4$. Otherwise you need some machine to do factoring approximation for you.

Theorem 5.2.9 *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

PROOF. Let $A \in \mathcal{M}_{n\times n}$ be triangular. Then the matrix $A - \lambda I$ is also triangular. For a triangular matrix, we can compute the determinant by multiplying the entries on the diagonal. Thus, if $a_1, a_2, \ldots, a_n$ are the entries on the diagonal of A, the diagonal of $A - \lambda I$ is $(a_1 - \lambda), (a_2 - \lambda), \ldots, (a_n - \lambda)$. So $\det(A - \lambda I) = (a_1 - \lambda)(a_2 - \lambda)\cdots(a_n - \lambda)$, and its zeros are exactly the entries on the diagonal of A. $\square$

Exploration 143 We know from Theorem 5.2.9 that the eigenvalues of

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

should be a, d, and f. Verify this by finding $\det(A - \lambda I)$, where $\lambda = a, d, f$.

At this point, we've defined all the eigenstuff[19] and developed a variety of methods for finding both eigenvalues and eigenvectors. We've proven our supposedly useful theorem about eigenvectors for different eigenvalues being linearly independent. What's left? More terminology, you say? Why, of course, if you insist.

19: What a fun word!
It's not a real word.

Definition 5.2.4 *For an eigenvalue λ of a matrix $A \in \mathcal{M}_{n\times n}$, the* ***algebraic multiplicity*** *of λ is the multiplicity of λ as a root of the characteristic polynomial for A. The* ***geometric multiplicity*** *of λ is the dimension of the eigenspace corresponding to λ.*

The multiplicity of λ as a root of a polynomial, $p(x)$, is then number of times $x - \lambda$ appears in the factorization of $p(x)$. For example, the multiplicity of 3 as a roof of $x^2 - 6x + 9$ is two because $x^2 - 6x + 9 = (x - 3)^2$.

Algebraic and geometric multiplicity give us a language to help organize the information about a matrix's eigenvalues and eigenvectors.

Example 5.2.7 Let

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Since A is triangular, we know its only eigenvalue is 5. Moreover, the characteristic polynomial is $(5 - \lambda)^3$, so the eigenvalue 5 has algebraic multiplicity of 3. Computing the geometric multiplicity takes a bit more work. We need to find the dimension of $\mathrm{Ker}\,(A - 5I)$.

$$A - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Oh, well maybe it's not much work in this case. This matrix has exactly two pivots, so the dimension of the kernel is one by the Rank-Nullity Theorem.[20] Thus, the geometric multiplicity of the eigenvalue 5 is 1.

20: Thank you, Ronnie Rank and Noether Nullity!

It would be really great if algebraic and geometric multiplicities were the same all the time.[21] It would also be great if the world was all candy canes and choo-choo trains, where the children of tomorrow dream away.[22]

21: Clearly this isn't the case. Besides the evidence of the previous example, why would there be two types of multiplicities with distinct names if they always agreed?

22: It's not.

Example 5.2.8 Let

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

By Theorem 5.2.9, 3 is the only eigenvalue of both A and B. For both matrices, the characteristic polynomial is $(\lambda - 3)^2$, so the algebraic multiplicity of the eigenvalue 3 is 2. However, it's easy to check that $\dim \operatorname{Ker}(A - 3I) = 1$, and $\dim \operatorname{Ker}(B - 3I) = 2$. Thus, the geometric multiplicity of the eigenvalue 3 for A is 1, and the geometric multiplicity of the eigenvalue 3 for B is 2.

You probably noticed that A and B are row equivalent. Yeah, that's right, they're row equivalent but have different eigenspaces. *Row operations mess up your eigenspace*. Fun fact: both A and B are also row equivalent to the identity, which doesn't even have the same eigenvalues, let alone eigenspaces. How messed up is that?

Exploration 144 Compare the eigenspaces of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -2 & -3 & 8 \\ 1 & 2 & -3 \\ -5 & -5 & 9 \end{bmatrix}.$$

Hint: They have the same eigenvalues.

Maybe you noticed that A and B are both invertible, so they're row equivalent matrices since both are row equivalent to I_3. What are we to conclude from this? Sometimes matrices with the same eigenvalues have the same geometric multiplicity and sometimes they don't? That is very dissatisfying. If only there were a way to classify when this happens? Clearly, row equivalence is not the right condition, but what could this condition be?[23]

23: Stay tuned for next week's episode, when our heroes Ronnie and Noether unravel the mystery of the unmatched algebraic and geometric multiplicities!

Section Highlights

- The determinant of a square matrix is a real number associated to that matrix that is computed through a recursive algorithm. See Definition 5.2.1, Example 5.2.1 and Theorem 5.2.1.
- A matrix is invertible if and only if its determinant is nonzero. See Theorem 5.2.5.
- If a matrix is diagonal, upper triangular, or lower triangular, then the determinant is the product of the diagonal entries, and the entries on

the diagonal are the eigenvalues. See Theorem 5.2.2 and Theorem 5.2.9.

- The characteristic polynomial of a matrix A is $\det(A-xI)$. Its zeros are the eigenvalues of A. See Definition 5.2.3 and Theorem 5.2.8.
- Every eigenvalue has an algebraic multiplicity (number of times it is the zero of the characteristic polynomial) and geometric multiplicity (dimension of the kernel of the associated eigenspace). See Definition 5.2.4.

Exercises for Section 5.2

5.2.1. Find the determinants of the matrices below.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -4 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}$

5.2.2. Use determinants to determine whether each of the following sets are linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

5.2.3. Let $A \in \mathcal{M}_{n\times n}$ and $k \in \mathbb{R}$. Find a formula for $\det(kA)$.

5.2.4. Let $A, B \in \mathcal{M}_{n\times n}$. Show that while AB may or may not be equal to BA, it is always the case that $\det(AB) = \det(BA)$.

5.2.5. Let $A \in \mathcal{M}_{n\times n}$ be such that $A^TA = I_n$. Show that either $\det A = 1$ or $\det A = -1$.

5.2.6. Let $A, B \in \mathcal{M}_{4\times 4}$ be such that $\det A = -1$ and $\det B = 2$. Calculate any of the following that can be calculated with the given information:

(a) $\det(4A^T)$

(b) $\det(A + B)$

(c) $\det(ABA^{-1})$

(d) $\det B^3$

5.2.7. Find the eigenvalues and corresponding eigenspaces for the following matrices. Also state the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

5.2.8. Bubbles was taking the train down to Gorky Park. A fellow passenger named Klaus claimed that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. Decide if this is nonsense; then prove it or provide a counterexample.

5.2.9. Show that 0 is an eigenvalue of $A \in \mathcal{M}_{n\times n}$ if and only if A is not invertible.

5.2.10. Let

$$A = \begin{bmatrix} 13 & 113/2 & 113 & 25/2 \\ -50 & -251 & -478 & -25 \\ 25 & 235/2 & 241 & 25/2 \\ -25 & -253/2 & -253 & -49/2 \end{bmatrix}.$$

Check that $\vec{x} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix}$ is an eigenvector for A. Use this fact to calculate $A^{10}\vec{x}$. What is $\lim_{n\to\infty} A^n\vec{x}$?

5.2.11. If $\vec{v}$ is an eigenvector of both A and B, show it is an eigenvector of AB and $A + B$.

5.3 Diagonalization

In the last section we talked about this eigenstuff[24] for matrices. However, we all know by now that matrices are inextricably linked to linear transformations between vector spaces. Let's talk a bit about eigenstuff in the context of linear transformations.

24:  Still not a word.

Linear Transformations and Invariant Subspaces

Suppose $T\colon V \to V$ is a linear transformation from an n-dimensional vector space V to itself, and let $A \in \mathcal{M}_{n\times n}$ be the matrix representing this linear transformation with respect to some basis $\mathcal{B}$ for V. Suppose λ is an eigenvalue for A with eigenvector $\vec{w}_\lambda \in \mathbb{R}^n$. Then there is a vector $\vec{v}_\lambda \in V$ for which $\vec{w}_\lambda$ is the coordinate vector with respect to $\mathcal{B}$. Thus, we know $T(\vec{v}_\lambda) = \lambda\vec{v}_\lambda$. Since T is a linear transformation, we also know

$$T(a\vec{v}_\lambda) = aT(\vec{v}_\lambda) = a\lambda(\vec{v}_\lambda) = \lambda(a\vec{v}_\lambda)$$

for any $a \in \mathbb{R}$. This says $\operatorname{Span}\{\vec{v}_\lambda\}$ is *preserved* by the linear transformation T. That is, $T(\vec{x}) \in \operatorname{Span}\{\vec{v}_\lambda\}$ for any $\vec{x} \in \operatorname{Span}\{\vec{v}_\lambda\}$. In this situation, we call $\operatorname{Span}\{\vec{v}_\lambda\}$ an *invariant subspace* of V for T. Perhaps that should be a definition.

Definition 5.3.1 *Let $T\colon V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a subspace of V. We say W is an* ***invariant subspace of** V **for** T if for any $\vec{x} \in W$, the vector $T(\vec{x})$ is also in W.*

In the discussion preceding the definition, we motivated this topic with its connection to eigenvectors, but that is not the only way these occur. It is quite possible to have invariant subspaces unrelated to eigenvectors.

Example 5.3.1 Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 + x_4 \\ x_3 + x_4 \end{bmatrix}.$$

Then $W = \operatorname{Span}\{\vec{e}_1, \vec{e}_2\}$ and $U = \operatorname{Span}\{\vec{e}_3, \vec{e}_4\}$ are invariant subspaces of $\mathbb{R}^4$ for T. Let's verify this for W together. Note first that $T(\vec{e}_1) = \vec{e}_2 \in W$ and $T(\vec{e}_2) = \vec{e}_1 \in W$. Since these form a basis for W, this is actually enough to show the entire subspace is invariant. To see this, note that any element of W is of the form $a\vec{e}_1 + b\vec{e}_2$ for some $a, b \in \mathbb{R}$, and

$$T(a\vec{e}_1 + b\vec{e}_2) = a\vec{e}_2 + b\vec{e}_1 \in W.$$

Exploration 145 Check that U is an invariant subspace as claimed.

There are very practical reasons to be aware of invariant subspaces in general. For instance, we mentioned in Chapter 1 that we can use subspaces containing all of our relevant data to reduce the size of the vector space we consider. Now, we can use invariant subspaces to reduce the "size" of linear transformations, while continuing to work in the current square matrix setting.

Definition 5.3.2 *Let $T\colon V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a subspace of V. The map*

$$T|_W : W \to V \text{ given by } T|_W(\vec{w}) = T(\vec{w})$$

for all $\vec{w} \in W$ is a map called the ***restriction of T to*** *W.*

Theorem 5.3.1 *Let $T\colon V \to V$ be a linear transformation from a vector space V to itself, and suppose W is a subspace of V. The restriction of T to W is a linear transformation. Moreover, if W is an invariant subspace of V for T, then $T|_W : W \to W$.*

This fact can quickly be proven since the linearity properties for T imply them for $T|_W$, so we will leave this for the exercises. Note that while we can always restrict ourselves to a subspace of the domain and define such a linear transformation, this is particularly nice when the subspace is invariant since this again gives us a linear transformation that can be represented by a square matrix.

Example 5.3.2 Suppose $T\colon V \to V$ is a linear transformation, V with basis $\mathcal{B}_V = \{\vec{b}_1, \ldots, \vec{b}_{7,000}\}$, and W is an invariant subspace of V for T with basis $\mathcal{B}_W = \{\vec{b}_1, \vec{b}_2\}$. Note first that the matrix representation for T will be $7,000 \times 7,000$, which isn't actually that large in practice (using computers), but it's certainly too large to write in this book. If $T(\vec{b}_1) = \vec{b}_1$, and $T(\vec{b}_2) = \vec{b}_1 + \vec{b}_2$, then the matrix representation with respect to $\mathcal{B}_W$ for $T|_W$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which doesn't even spill into the margin.[25]

25: Hey! You stay out of the margin!

Now, our goal was to discuss linear transformations and eigenstuff, so how does this fit in? Well, eigenspaces give us examples of invariant subspaces.

Theorem 5.3.2 *Suppose $T\colon V \to V$ is a linear transformation with matrix representation $A \in \mathcal{M}_{n\times n}$ with respect to the basis $\mathcal{B}$ of V for T. Suppose W is a subspace of V with a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ such that $[\vec{v}_i]_{\mathcal{B}}$ is an eigenvector for A for each $1 \le i \le k$. Then W is an invariant subspace of T.*

PROOF. Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is the described basis of W, and suppose they have eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$, respectively. If $\vec{w} \in W$, then $\vec{w} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ for some $a_1, \ldots, a_k \in \mathbb{R}$. We know that $[T(\vec{x})]_{\mathcal{B}} = A[\vec{x}]_{\mathcal{B}}$, and thus, if $[\vec{x}]_{\mathcal{B}}$ is an eigenvector for A with eigenvalue λ, then $T(\vec{x}) = \lambda\vec{x}$. Therefore,

$$T(\vec{w}) = T(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = a_1\lambda_1\vec{v}_1 + \cdots + a_k\lambda_k\vec{v}_k \in W$$

and W is preserved by T. $\square$

Since any eigenspace is made up entirely of eigenvectors, this implies any eigenspace for the matrix representation corresponds to an invariant subspace for the linear transformation. Thus, eigenvalues and eigenvectors provide an incredibly convenient way to find invariant subspaces.

Example 5.3.3 Define $T\colon \mathbb{P}_1 \to \mathbb{P}_1$ by

$$T\left(a+bx\right) = 2b + 2ax.$$

Then the matrix A for T using the basis $\{1, x\}$ for $\mathbb{P}_1$ is

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Let's find an eigenvalue and an eigenvector for this matrix.

$$\det(A - \lambda I) = \lambda^2 - 4$$
$$\lambda = -2, 2$$

For $\lambda = 2$, we have

$$A - 2I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

By computing the kernel of $A - 2I$ from this row reduction, we see that

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector for A corresponding to the eigenvalue 2.[26] Now, let's go back to our linear transformation T. The eigenvector we found corresponds to the polynomial $1 + x$. By our defining description of T, we have

$$T(1+x) = 2 + 2x = 2(1+x).$$

Aha! The "specialness" of 2 and $\vec{v}_2$ is present in the linear transformation T, not just it's matrix representation A. This says $\mathrm{Span}\,\{1+x\}$ is an invariant subspace of $\mathbb{P}_1$ for the linear transformation T.

What happens if we have a different matrix representation for T? If we consider a different basis for $\mathbb{P}_1$, will we still have this property? Let's see! Let $\mathcal{B} = \{1-x, 1+x\}$. This is a new basis[27] for $\mathbb{P}_1$. Since

$$T\left(1-x\right) = -2+2x = -2(1-x) \quad \text{and} \quad T\left(1+x\right) = 2+2x = 2(1+x)$$

we see the matrix B which represents T on this basis is

$$B = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Because this matrix is diagonal, we know it's eigenvalues are the entries on the diagonal 2 and -2, the same as those for A.[28] Now, let's consider the eigenvectors for B. Repeating our procedure of computing the kernel of $(B - \lambda I)$ from above we see that

$$\text{for } \lambda = -2: \quad \vec{v}_{-2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and for } \lambda = 2: \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

26: Did you catch what just happened? Did we just review all the procedures from the previous section? Why, yes, now that you mention it, we did. Isn't reviewing fun?

27: In the spirit of review, could you show this?

28: Wow, the review topics just keep on coming!

Okay, these aren't the same eigenvectors as we had for A, even though these two matrices did have the same eigenvalues. However, what we should really be writing is this:

$$\text{for } \lambda = -2 : \quad [\vec{v}_{-2}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and for } \lambda = 2 : \quad [\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since B is with respect to the basis $\mathcal{B}$, the eigenvectors for it are also with respect to the basis $\mathcal{B}$. Note that

$$[\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ means } \vec{v}_2 = 0(1-x) + 1(1+x) = 1 + x.$$

This agrees with the invariant subspace of $\mathbb{P}_1$ we found using the matrix A!

The example above illustrates that the eigenspaces of a matrix can be used to determine an invariant subspace for the linear transformation that corresponds to that matrix. Also, note how nice that diagonal matrix B looked. We'll talk more about this later. Lastly, these two matrices representing the same linear transformation had the same eigenvalues. Well, that's true in general. However, it's a bit easier to see if we momentarily forget about our linear transformation and focus specifically on matrices.

Similar Matrices

The matrices in the example above have the property that they both represent the same linear transformation, just with respect to different bases. Well, considering what we know from Section 4.6 about changing the basis, we have the following definition.

Definition 5.3.3 *Matrices $A, B \in \mathcal{M}_{n\times n}$ are **similar** if there is an invertible matrix $P \in \mathcal{M}_{n\times n}$ such that $A = PBP^{-1}$, or equivalently, $B = P^{-1}AP$.*

As mentioned above, two square matrices are similar if and only if they are matrix representations for the same linear transformations, just with different bases. Now, we can prove all we observed in the previous example using this concept of similarity.

Theorem 5.3.3 *Similar matrices have the same characteristic polynomial.*

Exploration 146 Proof by exploration! Suppose A and B are similar, so there is an invertible matrix P such that $A = PBP^{-1}$. Convince yourself that $A - \lambda I = PBP^{-1} - \lambda PP^{-1}$. Use this to show that $A - \lambda I = P(B - \lambda I)P^{-1}$.

Use the last equation to show that $\det(A - \lambda I) = \det(B - \lambda I)$.

Corollary 5.3.4 *If A has an eigenvalue λ with algebraic multiplicity k and B is similar to A, then B has the same λ as an eigenvalue with algebraic multiplicity k as well.*

Example 5.3.4 Recall Exploration 144 from Section 5.1. The matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -3 & 8 \\ 1 & 2 & -3 \\ -5 & -5 & 9 \end{bmatrix}$$

have the same eigenvalues, each with the same multiplicities, both geometric and algebraic. This is not a coincidence; these matrices are similar by way of the matrix

$$P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

This is readily verified:

$$\begin{aligned} PBP^{-1} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 8 \\ 1 & 2 & -3 \\ -5 & -5 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} = A. \end{aligned}$$

Theorem 5.3.3, however, is a one-way street. Having the same characteristic polynomial does not guarantee similarity. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same characteristic polynomial, but they are not similar.

This next theorem seems like maybe it should have been stated last section, when the concepts of algebraic and geometric multiplicities were introduced. Our proof, however, relies on Corollary 5.3.4.

Theorem 5.3.5 *For any eigenvalue, the algebraic multiplicity is greater than or equal to the geometric multiplicity.*

PROOF. Suppose $A \in \mathcal{M}_{n\times n}$ has geometric multiplicity k for eigenvalue λ, and $\{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis for the eigenspace E_λ associated to λ. Let $\{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ be a basis for $E_\lambda^\perp$. Then $\{\vec{x}_1, \ldots, \vec{x}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ forms a basis of $\mathbb{R}^n$ since we know $\mathbb{R}^n = E_\lambda \oplus E_\lambda^\perp$. Consider the matrix $S = [\vec{x}_1 \cdots \vec{x}_k\, \vec{v}_{k+1} \cdots \vec{v}_n]$. The columns of this matrix are a basis and thus linearly independent, so the matrix is invertible. Then the first k columns of $B = S^{-1}AS$ are the vectors $\lambda\vec{e}_1, \ldots \lambda\vec{e}_k$ since the first k basis vectors are eigenvectors for the eigenvalue λ.[29] Because λ is our specific eigenvalue, let us consider the characteristic polynomial as a polynomial in the variable x instead. Then $(\lambda - x)^k$ divides the characteristic polynomial of B. This can be seen by realizing the first k columns of B are all zero except with λ on the diagonal, so computing the determinant of $(B - xI)$ will be straightforward using the cofactor method down the first k columns. Thus, λ has algebraic

29:  If this seems unbelievable, you should do an example to convince yourself.

multiplicity at least k for B. By Corollary 5.3.4, it follows that λ has algebraic multiplicity at least k for A as well. $\square$

Remember from Theorem 5.3.2 that the eigenspace of the matrix representation for a linear transformation is an invariant subspace for the linear transformation. Moreover, two matrices that both represent the linear transformation T with respect to different bases will have the same eigenvalues and their eigenspaces will identify the same invariant subspaces for T. In other words, similar matrices must have the same eigenvalues with the same geometric multiplicities. We could prove this result using Theorem 5.3.2, but we will give a proof here that is in line with our matrix results instead.[30]

30: Oh, I'm sad. I prefer the linear transformations.

You're in luck! The authors accidentally proved it both ways, so that proof is in the Appendix.

How do you accidentally do twice the amount of work?

(shrugs)

Theorem 5.3.6 *If $A \in \mathcal{M}_{n\times n}$ has an eigenvalue λ with geometric multiplicity k and $B \in \mathcal{M}_{n\times n}$ is similar to A, then B has λ as an eigenvalue with geometric multiplicity k as well.*

PROOF. Suppose $\{\vec{x}_1, \ldots, \vec{x}_k\}$ is a basis for the eigenspace corresponding to the eigenvalue λ of A. Since $A = PBP^{-1}$, we have $B = P^{-1}AP$, and as we've seen before, $B - \lambda I = P^{-1}(A - \lambda I)P$. Then for any $1 \le i \le k$, we have

$$\begin{aligned}
(B-\lambda I)P^{-1}\vec{x}_i &= [P^{-1}(A-\lambda I)P]P^{-1}\vec{x}_i \\
&= P^{-1}(A-\lambda I)\vec{x}_i && \text{since } PP^{-1} = I_n \\
&= P^{-1}\vec{0} && \text{since } \vec{x}_i \in \operatorname{Ker}(A-\lambda I) \\
&= \vec{0},
\end{aligned}$$

so $\mathcal{B} = \{P^{-1}\vec{x}_1, \ldots, P^{-1}\vec{x}_k\} \subset \operatorname{Ker}(B - \lambda I)$. Moreover, since P^{-1} is invertible, we know that $\mathcal{B}$ is a linearly independent set. Thus, the geometric multiplicity of λ for B is at least k. To see that it is exactly k, we could reverse this argument, starting with B rather than A, and conclude the geometric multiplicity for A must be at least that of B. Thus, they must be equal. $\square$

Diagonalization

Let's discuss an application of all this. Well, it will eventually be an application. Suppose T is a linear transformation with matrix representation A, and suppose you have a valid, nay,[31] an *important* reason to apply T to a vector many, many times. That is, suppose you would like to *iterate* T. You would like to find $T(\vec{x})$ for some vector $\vec{x}$ and then you would also like to find $T(T(\vec{x}))$. Then maybe you would like to find $T(T(T(\vec{x})))$. Well, since composition of linear transformations corresponds to matrix multiplication, this means you need to know how to compute A^k for $k \ge 1$. Now, let's connect this to the concept of similarity we've been spending so much time on in this section.

31: Neigh? Is one of the authors trolling us?

It says "nay."

I prefer trolls to horses.

. . .

Example 5.3.5 Suppose A is similar to B, so there is an invertible matrix P such that $A = PBP^{-1}$. Then for any integer $k \ge 1$, A^k is similar to B^k.

Behold:

$$\begin{aligned} A^k &= \left(PBP^{-1}\right)^k \\ &= \left(PBP^{-1}\right)\left(PBP^{-1}\right)\cdots\left(PBP^{-1}\right) \\ &= PB(P^{-1}P)B\cdots(P^{-1}P)BP^{-1} \;=\; PB^kP^{-1}. \end{aligned}$$

Great! This seems to be leading somewhere. If only we had some particularly useful format of a matrix that we could use when raising it to a power...

Definition 5.3.4 *A matrix of the form*

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \in \mathcal{M}_{n\times n}$$

is called a ***diagonal matrix****.*

Diagonal matrices have the convenient property that

$$D^k = \begin{bmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^k \end{bmatrix}$$

Similarity allows us to exploit this further; if P is invertible, D is diagonal, and $A = P^{-1}DP$, then as we saw in Example 5.3.5,

$$A^k = (P^{-1}DP)^k = P^{-1}D^kP.$$

Definition 5.3.5 *A matrix is called* ***diagonalizable*** *if it is similar to a diagonal matrix.*

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$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix},$$

find

$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}^k.$$

Where does one find such a matrix P? Eigenvectors! It's literally a change of basis matrix formed by eigenvectors.

Theorem 5.3.7 *A matrix $A \in \mathcal{M}_{n\times n}$ is diagonalizable if and only if A has n linearly independent eigenvectors.*

PROOF. Suppose A is diagonalizable, so there is an invertible matrix $P = [\vec{v}_1 \cdots \vec{v}_n]$ and a diagonal matrix D such that $A = PDP^{-1}$, or $AP = PD$. Note that

$$\begin{aligned} AP &= [A\vec{v}_1 \cdots A\vec{v}_n] \text{ and} \\ PD &= [d_{11}\vec{v}_1 \cdots d_{nn}\vec{v}_n]; \end{aligned}$$

since $AP = PD$, we have for all $1 \le j \le n$,

$$A\vec{v}_j = d_{jj}\vec{v}_j.$$

Thus, the jth column of P is an eigenvector corresponding to the jth entry on the diagonal of D. That is, every column of P is an eigenvector of A. Moreover, since P is invertible, its columns are linearly independent, so these eigenvectors must be linearly independent.

On the other hand, suppose A has n linearly independent eigenvectors. Then one can construct P and D (as above) and readily verify that $AP = PD$. $\square$

Corollary 5.3.8 *The matrix $A \in \mathcal{M}_{n\times n}$ is diagonalizable if and only if for every eigenvalue of A, its geometric multiplicity is equal to its algebraic multiplicity.*

Corollary 5.3.9 *A matrix $A \in \mathcal{M}_{n\times n}$ with n distinct eigenvalues is diagonalizable.*

Here's a procedure for diagonalizing a matrix. Given a matrix $A \in \mathcal{M}_{n\times n}$,

(a) Find the eigenvalues of A.

(b) Find n linearly independent eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_n$. If there are not n of them, then be very sad; A is not diagonalizable.

(c) Construct P from these eigenvectors: $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$.

(d) Construct D from the eigenvalues with the eigenvalues along the diagonal in the same order as the eigenvectors in P.

(e) Check that $AP = PD$ (same as $A = PDP^{-1}$).[32]

32: Is checking really part of the procedure? Or is it just good advice? I am glad that you are not a civil engineer.

Example 5.3.6 Let

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Is A diagonalizable? Indeed. One can check that

$$\left\{ \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace corresponding to the eigenvalue 5, and

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace corresponding to the eigenvalue -3. Then

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Is A diagonalizable?

Section Highlights

- Matrices A and B are *similar* if there is some change of basis matrix P such that $A = PBP^{-1}$. See Definition 5.3.3 and the discussion surrounding it. Since they only differ by the basis they are with respect to, A and B are both matrix representations for the same linear transformation.
- We say a matrix is *diagonalizable* if it is similar to a diagonal matrix. See Definition 5.3.5 and Definition 5.3.3. If matrices A and B are both matrix representations for the same linear transformation, we say A and B are similar.
- An $n \times n$ matrix A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ made of eigenvectors for A. See Theorem 5.3.7.
- To diagonalize a matrix, we use the change of basis matrix to convert from the standard basis to a basis of eigenvectors. See Example 5.3.6.
- If λ is an eigenvalue for A, the *algebraic multiplicity* of λ is the multiplicity of $(x - \lambda)$ in the characteristic polynomial for A, and the *geometric multiplicity* of λ is the dimension of $\operatorname{Ker}(A - \lambda I)$. It's always true that the geometric multiplicity is less than or equal to the algebraic multiplicity. See Definition 5.2.4 and Theorem 5.3.5.
- A matrix A is diagonalizable if and only if the geometric and algebraic multiplicities agree for every eigenvalue. See Corollary 5.3.8.

Exercises for Section 5.3

5.3.1. Let $P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, and $A = PDP^{-1}$. Calculate A^9.

5.3.2. Here's a pretty big matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 & 4 \\ 0 & 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Hey, at least it's triangular.

(a) Without calculation of any kind, what are the eigenvalues and their algebraic multiplicities?

(b) Without calculation of any kind, what are the upper and lower bounds on the geometric multiplicity of each eigenvalue?

(c) Calculate the geometric multiplicities of each eigenvalue. Is A diagonalizable?

5.3.3. With minimal calculation, determine whether each of the following matrices is diagonalizable.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5.3.4. Let

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that that $A = PDP^{-1}$.

5.3.5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & 2 & -3 \\ -1 & 0 & 1 & -2 \\ -2 & -1 & 0 & -1 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that that $A = PDP^{-1}$.

5.3.6. For each of the following pairs of matrices, determine whether or not the matrices are similar. If they are similar, find a matrix P such that $A = P^{-1}BP$.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5.3.7. Show that every eigenspace is an invariant subspace. That is, if $A \in \mathcal{M}_{n\times n}$ has an eigenvalue λ with eigenspace E, show that E is an invariant subspace of $\mathbb{R}^n$ for T_A.

5.3.8. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix}.$$

It turns out that $\lambda = 2$ is an eigenvalue for A. Let E be the eigenspace associated to 2, and find a matrix representation for $A|_E$.

5.4 Jordan Canonical Form

Perhaps you're sold at this point on the greatness of diagonalizability. Whether or not that's the case, just how special are these diagonalizable matrices? What about matrices that aren't diagonalizable? How miserably "not diagaonalizable" can a matrix be? We'll see now, that they're not too miserable at all!

A New Form

Up to this point, we've only used vector spaces over $\mathbb{R}$, meaning all the scalars we use in scalar multiplication have been real numbers. See Definition 1.1.2. Even when we used the complex numbers as a vector space, it was as a *real* vector space. See Exercise 1.1.3. That ends now!

WARNING ALERT CAUTION ALERT WARNING

For the rest of the book, we're going to be using $\mathbb{C}$ for our scalars. That's what the garish warning was all about. Just know that it's happening, and don't panic. Vector spaces over $\mathbb{C}$ have their charm. Also, we won't explicitly be using them until we get to the proofs later in this section, and we'll keep the examples and explorations as real as possible.[33]

We have a good idea at this point how diagonalizable matrices work and which square matrices are diagonalizable. Now it is time to deal with the rest of them. That's right, *the rest of them*. It turns out that even square matrices that aren't diagonalizable are *almost diagonalizable* in a very specific, consistent sense. Let's see an example first, an incredibly long but deeply inspiring example.

Example 5.4.1 Here's a square matrix with its eigenvalue and eigenvector:

$$A = \begin{bmatrix} 2 & -5 & -6 \\ -2 & 2 & -4 \\ 5 & 7 & 14 \end{bmatrix}, \qquad \lambda = 6, \qquad \vec{v}_0 = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}.$$

That's right. There's only one eigenvalue with algebraic multiplicity 3 and geometric multiplicity 1. If we wanted to diagonalize A, we would need three linearly independent eigenvectors to use as the columns of P so that $AP = PD$, where D is a diagonal matrix. Alas, we only have one eigenvector. This matrix is not even a little bit diagonalizable. Well...
maybe a little bit...
We're missing two eigenvectors to build the matrix P used to diagonalize A, but perhaps we can find two stand-ins. What's so great about $\vec{v}_0$, anyway? Since it's an eigenvector for $\lambda = 6$, we know $(A - 6I)\vec{v}_0 = \vec{0}$. What if we found a vector that *maps into* $\operatorname{Ker}(A - 6I)$? In particular, we'd like to find a vector, $\vec{v}_1$ such that

$$(A - 6I)\vec{v}_1 = \vec{v}_0.$$

Then $(A - 6I)\vec{v}_1 = \vec{v}_0 \neq \vec{0}$ but $(A - 6I)^2\vec{v}_1 = \vec{0}$. This new vector, $\vec{v}_1$, isn't in $\operatorname{Ker}(A - 6I)$ because $A - 6I$ doesn't map it to $\vec{0}$, but if you apply $A - 6I$ to $\vec{v}_1$ *twice*, you do get $\vec{0}$. Thus, $\vec{v}_1$ isn't an eigenvector for $\lambda = 6$, but it's "the next best thing." It *eventually* gets mapped to $\vec{0}$ by repeated applications of $A - 6I$.

33: Really?
Really.
But aren't we all imaginary?
...!
I don't know about you two, but I'm totally real.

How do you find $\vec{v}_1$? Just solve $(A-6I)\vec{x}=\vec{v}_0$ for $\vec{x}$. Neat, right? Oh, why should a solution to $(A-6I)\vec{x}=\vec{v}_0$ exist? Yeah. That's a really good question. For now, let's just enjoy that is does, and we'll state and prove a theorem about that later. While we're at it, $(A-6I)\vec{x}=\vec{v}_1$ also has solutions; we'll call the one we're gonna use $\vec{v}_2$. Specifically, here are the solutions we chose:

$$\vec{v}_2=\begin{bmatrix}-1\\-2\\2\end{bmatrix}\qquad \vec{v}_1=\begin{bmatrix}1\\3\\-3\end{bmatrix}.$$

Alright. What exactly do these vectors, $\vec{v}_1$ and $\vec{v}_2$, do? They both eventually map to $\vec{0}$ after repeated application of $A-6I$. In fact,

$$\vec{v}_2\xrightarrow{A-6I}\vec{v}_1\xrightarrow{A-6I}\vec{v}_0\xrightarrow{A-6I}\vec{0}$$

This is neat, but you're probably wondering what good this little sequence (sometimes called a *Jordan chain*) is. We have the following two convenient implications:

$$\begin{aligned}(A-6I)\vec{v}_1=\vec{v}_0 \quad&\Rightarrow\quad A\vec{v}_1=\vec{v}_0+6\vec{v}_1\\(A-6I)\vec{v}_2=\vec{v}_1 \quad&\Rightarrow\quad A\vec{v}_2=\vec{v}_1+6\vec{v}_2.\end{aligned}$$

Now, let's make a matrix $P=[\vec{v}_0\ \vec{v}_1\ \vec{v}_2]$. Then

$$\begin{aligned}AP &= A[\vec{v}_0\quad \vec{v}_1\quad \vec{v}_2]\\&=[A\vec{v}_0\quad A\vec{v}_1\quad A\vec{v}_2]\\&=[6\vec{v}_0\quad \vec{v}_0+6\vec{v}_1\quad \vec{v}_1+6\vec{v}_2]\\&=[\vec{v}_0\quad \vec{v}_1\quad \vec{v}_2]\begin{bmatrix}6&1&0\\0&6&1\\0&0&6\end{bmatrix}=P\begin{bmatrix}6&1&0\\0&6&1\\0&0&6\end{bmatrix},\end{aligned}$$

so

$$A=PJP^{-1},\quad \text{where}\quad J=\begin{bmatrix}6&1&0\\0&6&1\\0&0&6\end{bmatrix}.$$

You may have noticed that J is not diagonal, but it's *almost* diagonal (except for those 1's). Well, $\vec{v}_1$ and $\vec{v}_2$ are not eigenvectors, but we chose them because they were the next best thing.

We mentioned a term in that example that deserves its own definition box:

Definition 5.4.1 *If $A\in\mathcal{M}_{n\times n}(\mathbb{C})$ has eigenvalue λ with eigenvector $\vec{v}_0$, then a* ***Jordan chain*** *for λ is a set of vectors $S=\{\vec{v}_1,\dots\vec{v}_k\}$ for some $k<n$ such that*

$$\vec{v}_k\xrightarrow{A-\lambda I}\vec{v}_{k-1}\xrightarrow{A-\lambda I}\cdots\xrightarrow{A-\lambda I}\vec{v}_1\xrightarrow{A-\lambda I}\vec{v}_0\xrightarrow{A-\lambda I}\vec{0}$$

This almost-diagonal matrix with all the rogue 1's just above the diagonal is also going to come up a lot.

Definition 5.4.2 *A* ***Jordan block*** *is a square matrix whose entries are the same constant, $\lambda\in\mathbb{C}$, on the diagonal, 1 on each entry immediately above the diagonal, and zero elsewhere.*

Example 5.4.2 Here are some Jordan blocks:

$$J_1 = \begin{bmatrix} 4 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 5 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \qquad J_4 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}.$$

Using appropriately sized matrices $[\vec{0}]$, we can combine Jordan blocks into one big matrix, too:

$$\begin{bmatrix} J_1 & \vec{0} & \vec{0} & \vec{0} \\ \vec{0} & J_2 & \vec{0} & \vec{0} \\ \vec{0} & \vec{0} & J_3 & \vec{0} \\ \vec{0} & \vec{0} & \vec{0} & J_4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix},$$

Pretty close to diagonal, right?

Theorem 5.4.1 (Jordan Canonical Form) *Every square matrix, $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, is similar by $P \in \mathcal{M}_{n\times n}(\mathbb{C})$ to a matrix, $J \in \mathcal{M}_{n\times n}(\mathbb{C})$, whose only nonzero entries are Jordan blocks on the diagonal. Moreover, J is unique (allowing for the reordering of the Jordan blocks). The matrix J is called the Jordan Canonical Form of the matrix A.*

Perhaps at this point, you find this very believable. There are some things to prove, but let's wait a minute and play with this fun new theorem first.

Example 5.4.3

$$A = \begin{bmatrix} 11 & 5 & 2 & 0 & -2 & -7 & -3 \\ -5 & 3 & 1 & 0 & 1 & 4 & 3 \\ 14 & 13 & 11 & 0 & -5 & -19 & -7 \\ 2 & 0 & -3 & 4 & 0 & 1 & -2 \\ 1 & 3 & 3 & 0 & 4 & -4 & 0 \\ 8 & 8 & 4 & 0 & -3 & -6 & -4 \\ -9 & -8 & -3 & 0 & 3 & 12 & 10 \end{bmatrix},$$

$$\begin{aligned} P &= [\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4\ \vec{p}_5\ \vec{p}_6\ \vec{p}_7] \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}, \text{ and} \end{aligned}$$

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

You can check that $A = PJP^{-1}$ (or you can trust us). J is the Jordan Canonical Form for A. Here are some fun facts about eigenvalues and eigenvectors for A that you can check:

eigenvalue	4	5	6
alg. mult.	1	3	3
eigenvectors	$\vec{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$, $\vec{p}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\vec{p}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

This begs the question, where did $\vec{p}_4$, $\vec{p}_6$, and $\vec{p}_7$ come from? As in the example from the beginning of this section, we found the vectors $\vec{p}_4$, $\vec{p}_6$, and $\vec{p}_7$ so that

$$\begin{aligned} (A - 5I)\vec{p}_4 &= \vec{p}_3, \\ (A - 6I)\vec{p}_6 &= \vec{p}_5, \text{ and} \\ (A - 6I)\vec{p}_7 &= \vec{p}_6. \end{aligned}$$

Finding these vectors is as easy as solving these matrix equations. The remarkable thing about Theorem 5.4.1 is that is guarantees these vectors exist. We'll get into how this is done procedurally at the end of the section (after we prove the theorem).

Recall our heuristic for diagonalization. If an $n \times n$ matrix is diagonalizable, it's similar to a diagonal matrix, so the matrix is just scalar multiplication in n linearly independent directions. What are we to make of Jordan Canonical form? We need a couple more definitions.

Definition 5.4.3 *The* ***left shift*** *on $\mathbb{C}^n$ is the linear transformation $L \colon \mathbb{C}^n \to \mathbb{C}^n$ given by $L(x_1, \ldots, x_n) = (x_2, \ldots, x_n, 0)$.*

Or is it an *up* shift? Lots of people write their vectors vertically, so this linear transformation, even when written that way, is called the left shift. You should verify that the left shift is a linear transformation! Oh, hey; find its kernel and image, too![34]

34: Exercise!

Theorem 5.4.2 *As a linear transformation, any Jordan block is the sum of a scalar multiple of the identity map and the left shift.*

PROOF. Let $J \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a Jordan block, so for $\vec{x} \in \mathbb{C}^n$,

$$
\begin{aligned}
J\vec{x} &= \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda x_1 + x_2 \\ \lambda x_2 + x_3 \\ \lambda x_3 + x_4 \\ \vdots \\ \lambda x_{n-1} + x_n \\ \lambda x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \lambda x_{n-1} \\ \lambda x_n \end{bmatrix} + \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ 0 \end{bmatrix} = \lambda\vec{x} + L(\vec{x}),
\end{aligned}
$$

where L is the left shift. How very gaudy. We probably should've just done this with a 3×3 and asked you to use your imagination. □

Definition 5.4.4 *Let $T\colon V \to V$ be a linear transformation with invariant subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$. Then every element of $\vec{v} \in V$ can be written as $\vec{v} = \vec{v}_1 + \vec{v}_2$ for some $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$. Thus, we have*

$$T(\vec{v}) = T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = T|_{V_1}(\vec{v}_1) + T|_{V_2}(\vec{v}_2).$$

In this situation, we say T decomposes into the ***direct sum of linear transformations*** *$T|_{V_1}$ and $T|_{V_2}$ and denote this as $T = T|_{V_1} \oplus T|_{V_2}$.*

Note that we only defined this here for two invariant subspaces V_1 and V_2, but it's possible that the vector space $V = V_1 \oplus \cdots \oplus V_k$ for some $k \geq 2$, where each V_i is an invariant subspace. The definition makes sense in this setting as well.[35]

35: Really? Are you sure?

Exercise!

Now, we previously talked quite a bit about how two matrices are similar if they are each the matrix representation for the same linear transformation. Well, we can flip this as well, to talk about when two different linear transformations have a relationship given by their matrix representations.

Definition 5.4.5 *Two linear transformations are* ***similar linear transformations*** *if they have similar matrix representations.*

The following corollary nicely describes what Jordan Canonical Form actually does and is due to Terry Tao.

Corollary 5.4.3 *Every linear transformation is similar to a direct sum of linear transformations, where each summand is itself the sum of a scalar multiple of the identity map and the left shift.*

It's not *quite* as good as diagonalization, where a matrix is similar to a direct sum of scalar multiplication maps, but it's not too far off. It's just a little...*shiftier*.

Example 5.4.4 Let $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ by $T(c+bx+ax^2) = b+2ax$. The matrix representation for T using the basis $\{1, x, x^2\}$ is a matrix A, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can see that A is similar to a matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

since

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Note that $B = 0I_3 + L$, so T is similar to T_1, where $T_1(c+bx+ax^2) = b+ax = 0(c+bx+ax^2)+L(c+bx+ax^2)$, where the first term is a scalar multiple of the identity and the second is a left shift.

In Example 5.4.4, we identified two matrices were similar by finding the matrix P such that $B = P^{-1}AP$. Well, it can sometimes be difficult to find that matrix. However, Jordan Canonical Form gives us a systematic approach to identifying whether two matrices are similar.

Corollary 5.4.4 *Suppose $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $B \in \mathcal{M}_{n\times n}(\mathbb{C})$. Then A and B are similar if and only if they are both similar to the same matrix in Jordan Canonical Form.*

This follows from the fact that the Jordan Canonical Form is unique up to rearranging placements of the Jordan blocks along the diagonal and one can use similarity to permute blocks along the diagonal.

Proof for Jordan Canonical Form

We still have some proving to do. We should also talk about how one actually finds a Jordan Canonical Form in practice. Proofs first!

Remember the beginning of this section when we announced the arrival of complex numbers? Right. Allowing complex numbers gives us the following super-handy theorem.

Theorem 5.4.5 (The Fundamental Theorem of Algebra) *Every degree $n \geq 1$ polynomial in one variable with complex coefficients has exactly n complex roots (counting the multiplicity of the repeated roots).*

Proofs of the Fundamental Theorem of Algebra are readily available. We provide none of them here.

Maybe we should have mentioned this before, but there is a small cost associated to the switch to complex numbers. Almost everything we've done up to this point remains the same, but with complex numbers, we need to make a small change to the definition of inner product (and hence, norm).

Complex numbers are like sand; once you have some of them in a vector space, they get all over everything. Even just using complex scalars, we end up having to deal with complex vectors as well because of linear combinations with complex scalars. For example, if $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the linear combination

$$(1+i)\vec{x} + (2+4i)\vec{y}$$

is definitely *not* in $\mathbb{R}^n$. It is in $\mathbb{C}^n$, though. How is this a problem? Well, for $\vec{x} \in \mathbb{R}^n$, we defined $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$. What happens with this definition for a complex vector?

$$\begin{aligned} \left\| \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 1+i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1+i \\ 1 \end{bmatrix}} &= \sqrt{(1+i)(1+i)+1} \\ &= \sqrt{1+2i}. \end{aligned}$$

This is weird. You probably wouldn't be surprised to find that the square root of complex numbers is a complex number that is almost always not a real number; that is the case with $\sqrt{1+2i}$. If the norm of a vector is supposed to represent the *magnitude* of a vector, what are we to make of a complex magnitude? Right. We should fix that. The fix, it turns out, is pretty easy.

Definition 5.4.6 *For $z = x + iy \in \mathbb{C}$, the* ***conjugate of z*** *is the complex number $\bar{z} = x - iy$.*

Definition 5.4.7 *The* ***Hermitian Inner product*** *is the function $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by*

$$\vec{v} \cdot \vec{u} = v_1\bar{u}_1 + \cdots + v_n\bar{u}_n = \sum_{i=1}^{n} v_i\bar{u}_i.$$

The notation is the same as the real inner product because the Hermitian inner product on real vectors is just the usual inner product.[36] The key is simply to remember that when you have complex vectors to conjugate the entries of the second vector when doing an inner product. How does this help? It turns out that for any $z \in \mathbb{C}$, we have $z\bar{z} \in \mathbb{R}$.[37]

36: Check this!

37: You check this!

Definition 5.4.8 *The* ***Hermitian norm*** *is the function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ defined for any $\vec{v} \in \mathbb{C}^n$ as*

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1\bar{v}_1 + \cdots + v_n\bar{v}_n}.$$

Using the Hermitian inner product and norm is the only alteration we will need to make in this book when working with complex numbers. In the interest of full disclosure, here's a potentially upsetting fact; there are a lot of different ways we could've defined inner products. We chose some specific ones for simplicity's sake here and in earlier chapters.

Now, back to Jordan Canonical Form. We get to use the Fundamental Theorem of Algebra right away!

Lemma 5.4.6 *Suppose W_1 and W_2 are subspaces of vector space V with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that $W_1 \oplus W_2 = V$. Let A be the matrix representation of $T|_{W_1}$ with respect to $\mathcal{B}_1$, and B be the matrix representation of $T|_{W_2}$ with respect to $\mathcal{B}_2$. If W_1 and W_2 are both invariant for T, then*

$$\begin{bmatrix} A & \vec{0} \\ \vec{0} & B \end{bmatrix}$$

is the matrix representation of T with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2$. If just W_1 is invariant for T, then

$$\begin{bmatrix} A & R \\ \vec{0} & B \end{bmatrix}$$

is the matrix representation of T with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2$.

Note that if $\mathcal{B}_1 = \{\vec{x}_1, \ldots, x_k\}$, $\mathcal{B}_2 = \{\vec{y}_{k+1}, \ldots, \vec{y}_n\}$ and both W_1 and W_2 are invariant for T, then for any $\vec{x}_i \in W_1$ and $\vec{y}_i \in W_2$, we have

$$\begin{array}{rcccccccc} T(\vec{x}_i) & = & a_{i1}\vec{x}_1 & + \cdots + & a_{ik}\vec{x}_k & + & 0\vec{y}_{k+1} & + \cdots + & 0\vec{y}_n, \\ T(\vec{y}_i) & = & 0\vec{x}_1 & + \cdots + & 0\vec{x}_k & + & a_{i(k+1)}\vec{y}_{k+1} & + \cdots + & a_{ik}\vec{y}_n. \end{array}$$

The proof follows then from computing the matrix for T with respect to this basis $\mathcal{B}_1 \cup \mathcal{B}_2$. We leave this proof then as an exercise, but we will revisit a familiar example illustrating this.

Exploration 149 Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 + x_4 \\ x_3 + x_4 \end{bmatrix}.$$

In Example 5.3.1, we saw that T has two invariant subspaces $W = \text{Span}\{\vec{e}_1, \vec{e}_2\}$ and $U = \text{Span}\{\vec{e}_3, \vec{e}_4\}$. Compute the matrix for T with respect to the standard basis. Compare this to the form from Lemma 5.4.6.

Theorem 5.4.7 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, then there is an invertible matrix Q and an upper triangular matrix U such that $A = QUQ^{-1}$ and U has the eigenvalues of A on its diagonal.*

PROOF. By the Fundamental Theorem of Algebra, $\det(A - \lambda I) = 0$ has a solution, call it λ_1. By Theorem 5.2.8, λ_1 is an eigenvalue, so it has at least one eigenvector. Let $\{\vec{x}_1, \ldots, \vec{x}_j\}$ be a basis for E_1, the eigenspace for λ_1. Using this with a basis for $E_1^\perp$, $\{\vec{v}_{j+1}, \ldots, \vec{v}_n\}$, define the change of basis matrix, $Q_1 = [\vec{x}_1 \cdots \vec{x}_j \; \vec{v}_{j+1} \; \cdots \; \vec{v}_n]$. While E_1 is an invariant subspace, $E_1^\perp$ might not be,[38] so by Lemma 5.4.6

$$Q_1^{-1}AQ_1 = \begin{bmatrix} \lambda_1 I_j & R_1 \\ \vec{0} & C_1 \end{bmatrix},$$

where $R_1 \in \mathcal{M}_{1,n-j}(\mathbb{C})$, $\vec{0} \in \mathcal{M}_{n-j,1}(\mathbb{C})$, and $C_1 \in \mathcal{M}_{n-j,n-j}(\mathbb{C})$. This was a very nice and repeatable procedure.

38: Make an example in which $E^\perp$ is not invariant!

Again by the Fundamental Theorem of Algebra, C_1 has an eigenvalue, λ_2. Let $\{\vec{y}_1, \ldots, \vec{y}_k\}$ be a basis for E_2, the eigenspace for λ_2. Using this with a basis for $E_2^{\perp}$, $\{\vec{u}_{n-j-k}, \ldots, \vec{u}_n\}$, define the change of basis matrix $Q_2 = [\vec{e}_1, \ldots, \vec{e}_j \vec{y}_1 \; \cdots \; \vec{y}_k \; \vec{u}_{n-k-j} \; \cdots \; \vec{u}_n]$, where each $\vec{e}_i$ is a standard basis vector. Then

$$Q_2^{-1} Q_1^{-1} A Q_1 Q_2 = Q_2^{-1} \begin{bmatrix} \lambda_1 I_j & R_1 \\ \vec{0} & C_1 \end{bmatrix} Q_2 = \begin{bmatrix} \lambda_1 I_j & R_2 & R_3 \\ \vec{0} & \lambda_2 I_k & R_4 \\ \vec{0} & \vec{0} & C_2 \end{bmatrix},$$

where $C_3 \in \mathcal{M}_{n-j-k \times n-j-k}(\mathbb{C})$. Do this $n-2$ more times, and define $Q = Q_1 Q_2 \cdots Q_{n-1}$.

By the Fundamental Theorem of Algebra, $\det(A - \lambda I) = 0$ has n solutions, counting multiplicity, which are all eigenvalues; call them $\lambda_1, \ldots, \lambda_m$, and $\det(U - \lambda I) = \det(Q^{-1} A Q - \lambda I) = 0$ has the same n solutions, which are the eigenvalues of A. By construction, these eigenvalues are on the diagonal of U. □

Theorem 5.4.8 *Let $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ be upper triangular with diagonal entries d_i, for $i = 1, \ldots, n$. Then for any integer $k \geq 0$, U^k is upper triangular with diagonal entries d_i^k, for $i = 1, \ldots, n$. If the diagonal entries of U are all zeros, then for any $0 \leq k \leq n$, U^k will have zero for all of its diagonal entries and for all k entries above each diagonal entry.*

Proof for this theorem can be found in the Appendix.

Lemma 5.4.9 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ has eigenvalue, λ, with algebraic multiplicity k and geometric multiplicity j, then there is a positive integer $k_0 \leq k$ such that*

$$\dim \operatorname{Ker} (A - \lambda I)^{k_0} = k,$$

and for each $i = 1, \ldots, k_0 - 1$, there is an m_{i+1} such that $1 \leq m_{i+1} \leq k - j$,

$$\dim \operatorname{Ker} (A - \lambda I)^i + m_{i+1} = \dim \operatorname{Ker} (A - \lambda I)^{i+1},$$

and $j + m_2 + \cdots + m_{k_0} = k$.

Proof. Using Theorem 5.4.7, we have that A is similar to an upper triangular matrix, U, such that the diagonal contains the eigenvalues of A, counting multiplicity. Let's cleverly arrange that the λ's are the first k entries on the diagonal. If E is the eigenspace for λ, then

$$U|_E = \begin{bmatrix} \lambda & * & \cdots & \cdots & * \\ \vec{0} & \lambda & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vec{0} & \cdots & \vec{0} & \lambda & * \\ \vec{0} & \cdots & \cdots & \vec{0} & \lambda \end{bmatrix}, (U - \lambda I)|_E = \begin{bmatrix} \vec{0} & * & \cdots & \cdots & * \\ \vec{0} & \vec{0} & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vec{0} & \cdots & \vec{0} & \vec{0} & * \\ \vec{0} & \cdots & \cdots & \vec{0} & \vec{0} \end{bmatrix}.$$

By Theorem 5.4.8, we know that $(U - \lambda I)|_E^{k_0} = \vec{0}$, the zero matrix. Since the last $n-k$ diagonal entries of U are different from λ, the last $n-k$ entries of $U - \lambda I$ must be nonzero. Thus, $\dim \operatorname{Ker} (U - \lambda I)^{k_0} = k$. Since $(U - \lambda I)^{k_0}$ is similar[39] to $(A - \lambda I)^{k_0}$, we know from Theorem 5.3.6 that $\dim \operatorname{Ker} (A - \lambda I)^{k_0} = k$ as well. Moreover, at each step from $(U - \lambda I)^i$ to $(U - \lambda I)^{i+1}$

39:  Do you remember why? If not, can you verify the calculation?

for $I = 1, \dots, k_0 - 1$, at least one more free variable is produced, so we also have that $\dim \operatorname{Ker}(A - \lambda I)^i + 1 \leq \dim \operatorname{Ker}(A - \lambda I)^{i+1}$. $\square$

That "at least one more free variable" bit is annoying. Look at this matrix and its square:

$$B - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (B - 2I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $B - 2I$ has two free variables, but $(B - 2I)^2$ has four; $\dim \operatorname{Ker}(B - 2I)^2$ has increased by two. What is happening here? It turns out that the eigenvalue 2 has two distinct Jordan chains corresponding to two distinct eigenvectors, $\vec{e}_1$ and $\vec{e}_4$:

$$\begin{array}{ccccccc} \vec{e}_3 & \xrightarrow{B-2I} & \vec{e}_2 & \xrightarrow{B-2I} & \vec{e}_1 & \xrightarrow{B-2I} & \vec{0} \\ & & \vec{e}_5 & \xrightarrow{B-2I} & \vec{e}_4 & \xrightarrow{B-2I} & \vec{0}. \end{array}$$

Here $\{\vec{e}_1, \vec{e}_4\}$ is a basis for $\operatorname{Ker}(B - 2I)$, and $\{\vec{e}_1, \dots, \vec{e}_5\}$ is a basis for $\operatorname{Ker}(B - 2I)^3$.

Here, finally, is the formal statement of the fact that we can find nice substitutes for eigenvectors, like we did in Example 5.4.1. If a matrix has an eigenvalue with algebraic multiplicity k and geometric multiplicity j, then you can always find j Jordan chains with a combined total length of k. Here's the more formal statement:

Lemma 5.4.10 *Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ have an eigenvalue, λ, with algebraic multiplicity k and geometric multiplicity j. Then there are j Jordan chains, $S_1, \dots, S_j$, such that $S_1 \cup \cdots \cup S_j$ is a basis for* $\operatorname{Ker}(A - \lambda I)^k$, *and in particular,*

$$\operatorname{Ker}(A - \lambda I)^k = \operatorname{Span}\{S_1\} \oplus \cdots \oplus \operatorname{Span}\{S_j\}.$$

It's not hard to believe there are j Jordan chains because there are j distinct eigenvectors that form a linearly independent set. It is definitely worth checking, though, that the associated *Jordan chains* are also formed of vectors that collectively form a basis for $\operatorname{Ker}(A - \lambda I)^k$.

There is a lot of bookkeeping and paperwork in the proof for this lemma. The process is neither elegant nor enlightening, so we relegate most of the fine details to the Appendix. For now, we'll provide a sketch of how the proof works.

Our goal is to first argue that these j Jordan chains exist and are distinct, and then argue that they form a basis for $\operatorname{Ker}(A - \lambda I)^k$. Once they form a basis, we know we can rearrange these basis vectors as the Jordan chains, $S_1, \dots, S_j$, to get

$$\operatorname{Ker}(A - \lambda I)^k = \operatorname{Span}\{S_1\} \oplus \cdots \oplus \operatorname{Span}\{S_j\}.$$

As was stated earlier, though, we begin with the argument that these distinct chains exist. We'll use the same notation as Lemma 5.4.9, noting that for some

$k_0 \leq k$, we have Ker $(A - \lambda I)^k = \text{Ker}\,(A - \lambda I)^{k_0}$. One can also check that

$$\text{Ker}\,(A - \lambda I) \subset \text{Ker}\,(A - \lambda I)^2 \subset \cdots \subset \text{Ker}\,(A - \lambda I)^{k_0}.$$

Using the Orthogonal Decomposition Theorem several times,[40] we get a nice break down of Ker $(A - \lambda I)^{k_0}$ into orthogonal parts. Specifically, we can write

$$\begin{array}{rcll} E_1 & = & \text{Ker}\,(A - \lambda I), & \dim E_1 = j \\ E_2 & = & (E_1)^{\perp} \cap \text{Ker}\,(A - \lambda I)^2, & \dim E_2 = m_2 \\ E_3 & = & (E_1 \oplus E_2)^{\perp} \cap \text{Ker}\,(A - \lambda I)^3, & \dim E_3 = m_3 \\ & \vdots & & \\ E_{k_0} & = & (E_1 \oplus \cdots \oplus E_{k_0 - 1})^{\perp}, & \dim E_{k_0} = m_{k_0}. \end{array}$$

From this construction, we have

$$\begin{array}{rcl} \text{Ker}\,(A - \lambda I) & = & E_1 \\ \text{Ker}\,(A - \lambda I)^2 & = & E_1 \oplus E_2 \\ \text{Ker}\,(A - \lambda I)^3 & = & E_1 \oplus E_2 \oplus E_3 \\ & \vdots & \\ \text{Ker}\,(A - \lambda I)^{k_0} & = & E_1 \oplus \cdots \oplus E_{k_0} \end{array}$$

Now we can build all of our chains, all at once. Starting with a basis for E_{k_0}, we multiply each of these basis vectors by $(A - \lambda I)$ and extend the resulting set of vectors to make a basis for $E_1 \oplus \cdots \oplus E_{k_0 - 1}$. Next, we multiply some of the vectors in *that* basis by $(A - \lambda I)$ and again extend the resulting set of vectors to be a basis for $E_1 \oplus \cdots \oplus E_{k_0 - 2}$. We keep doing this until finally arriving in E_1. Not so bad, right? At each step, however, we have to argue that our sets of vectors are linearly independent so that we know the chains remain distinct; this is aided by the use of orthogonal complements to define each of the sets, E_i. The result will be a basis for E_1 built out of the final vectors from each of the Jordan chains (some of which will be of length one). This construction gives us our j distinct Jordan chains. Then we only need to argue that any two Jordan chains are linearly independent, so that the collection of all the chains forms a basis for Ker $(A - \lambda I)^k$ as desired.

Since we will use these Jordan chains to form a basis of our domain $\mathbb{C}^n$, we need to know that they are also linearly independent when they correspond to different eigenvalues.

Lemma 5.4.11 *If S_1 and S_2 are Jordan chains for $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ for different eigenvalues, then $S_1 \cup S_2$ is linearly independent.*

The proof of this is similar to the one for Theorem 5.1.2, so we've put it as an exercise. Now, for one final lemma before we prove the theorem.

Lemma 5.4.12 *Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ have an eigenvalue, λ, with Jordan chain S of length k, the algebraic multiplicity of λ. Then the matrix representation with respect to S for A is a Jordan block.*

This proof is very similar to the argument in Example 5.4.1, so this is also an exercise.

40: Is it exactly $k_0 - 1$ times?

It's cheating if you read ahead.

PROOF OF THEOREM 5.4.1. Applying Lemma 5.4.10 to all the eigenvalues of A, we have a set of Jordan chains with n vectors, and by Lemma 5.4.11, they are all linearly independent; thus, this set is a basis for $\mathbb{C}^n$, call it $\mathcal{B}$. Once we show the span of each chain, $\operatorname{Span}\{S_i\}$, from Lemma 5.4.10 is invariant for A, we can apply Lemmas 5.4.6 and 5.4.12 to each $\operatorname{Span}\{S_i\}$, and we're done.

Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a Jordan chain for eigenvalue λ and $\vec{v} \in \operatorname{Span}\{S\}$. Then $(A - \lambda I)\vec{v}_i = \vec{v}_{i-1}$ for each $i = 2, \dots, k-1$, and $(A - \lambda I)\vec{v}_1 = \vec{0}$; it follows that $A\vec{v}_i = \lambda\vec{v}_i + \vec{v}_{i-1}$, and $A\vec{v}_1 = \lambda\vec{v}_1$. Moreover, $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ for some scalars $a_1, \dots, a_n$. Observe that

$$\begin{aligned}
A\vec{v} &= A(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \\
&= a_1(\lambda\vec{v}_1) + a_2(\lambda\vec{v}_2 + \vec{v}_1) + \cdots + a_n(\lambda\vec{v}_n + \vec{v}_{n+1}) \\
&= (a_1\lambda + a_2)\vec{v}_1 + (a_2\lambda + a_3)\vec{v}_2 + \cdots + (a_{n-1}\lambda + a_n)\vec{v}_{n-1} + (a_n\lambda)\vec{v}_n,
\end{aligned}$$

which is a vector in $\operatorname{Span}\{S\}$, so $\operatorname{Span}\{S\}$ is invariant.

To see that the Jordan form is unique up to rearranging the blocks on the diagonal, note that the sizes of the blocks correspond to the lengths of the Jordan chains. These in turn were determined by the dimensions of each relevant $\operatorname{Ker}(A - \lambda I)^l$ for the eigenvalues λ. Since these are intrinsic to the original matrix and did not rely upon choices, the form is unique. $\square$

Alright! Victory!

Computing Jordan Canonical Form

The procedure for finding Jordan Canonical Form is actually pretty thoroughly described in Example 5.4.1. We'll just go through one more example in detail and then call it a day.

Example 5.4.5 Consider the matrix

$$A = \begin{bmatrix} 5 & 2 & -1 & 0 & 2 \\ -2 & 1 & 2 & 1 & -3 \\ -1 & -1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 4 \end{bmatrix}.$$

We can use techniques from the previous section to find the characteristic polynomial. Of course, then it's a degree 5 polynomial that needs factoring, so we'll just go ahead and tell you the characteristic polynomial here is $p(\lambda) = (3 - \lambda)^5$. Thus, the only eigenvalue is 3, and it has an algebraic multiplicity of 5. Let's compute $\operatorname{Ker}\ A - 3I$ to see the geometric multiplicity.

$$A - 3I = \begin{bmatrix} 2 & 2 & -1 & 0 & 2 \\ -2 & -2 & 2 & 1 & -3 \\ -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\mathrm{Ker}\,(A-3I)=\mathrm{Span}\left\{\vec{v}_1=\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix},\vec{v}_2=\begin{bmatrix}-1\\0\\0\\1\\1\end{bmatrix}\right\}.$$

Since the geometric multiplicity is 2, we need 2 Jordan chains. To get the next step, we need to solve $(A-3I)\vec{v}_3=\vec{v}_1$ and also $(A-3I)\vec{v}_4=\vec{v}_2$. We can do this more efficiently by augmenting with two vectors and performing the same row reduction steps as we used above.

$$\left[\begin{array}{rrrrr|rr} 2 & 2 & -1 & 0 & 2 & -1 & -1\\ -2 & -2 & 2 & 1 & -3 & 1 & 0\\ -1 & -1 & -1 & -1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$
$$\to\left[\begin{array}{rrrrr|rr} 1 & 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1 & -1 & -1 & -4\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This tells us that

$$\begin{aligned}\vec{v}_3 &\in \left\{\begin{bmatrix}-x_2-x_5\\x_2\\1\\-1+x_5\\x_5\end{bmatrix}: x_2,x_5\in\mathbb{R}\right\}\text{ and}\\ \vec{v}_4 &\in \left\{\begin{bmatrix}1-x_2-x_5\\x_2\\3\\-4+x_5\\x_5\end{bmatrix}: x_2,x_5\in\mathbb{R}\right\}.\end{aligned}$$

We have infinitely many choices here for $\vec{v}_3$ and $\vec{v}_4$, but we can simply choose that $x_2=x_5=0$ in both sets to get

$$\vec{v}_3=\begin{bmatrix}1\\0\\3\\-4\\0\end{bmatrix}\quad\text{and}\quad\vec{v}_4=\begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}.$$

Now, since we need just one more vector to form a basis, we know only one of these chains continues. However, it's not obvious which one it is. We can again augment by two vectors and solve simultaneously for $(A-3I)\vec{v}_5=\vec{v}_3$

and $(A-3I)\vec{v}_6 = \vec{v}_4$, knowing that only one of these has a solution.

$$\left[\begin{array}{rrrrr|rr} 2 & 2 & -1 & 0 & 2 & 1 & 0 \\ -2 & -2 & 2 & 1 & -3 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 3 & 1 \\ 1 & 1 & 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|rr} 1 & 1 & 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 & 0 & -9 & -2 \\ 0 & 0 & 0 & 1 & -1 & 10 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

This says $\vec{v}_5$ does not exist and

$$\vec{v}_6 \in \left\{ \begin{bmatrix} -1 - x_2 - x_5 \\ x_2 \\ -2 \\ 2 + x_5 \\ x_5 \end{bmatrix} : x_2, x_5 \in \mathbb{R} \right\}.$$

Again, we can choose that $x_2 = x_5 = 0$, so that

$$\vec{v}_6 = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 2 \\ 0 \end{bmatrix}.$$

Now, our Jordan chains are

$$\begin{array}{ccccccc} \vec{v}_6 & \xrightarrow{A-3I} & \vec{v}_4 & \xrightarrow{A-3I} & \vec{v}_2 & \xrightarrow{A-3I} & \vec{0} \\ & & \vec{v}_3 & \xrightarrow{A-3I} & \vec{v}_1 & \xrightarrow{A-3I} & \vec{0}, \end{array}$$

and our basis for Jordan Canonical Form is $\{\vec{v}_1, \vec{v}_3, \vec{v}_2, \vec{v}_4, \vec{v}_6\}$.

Exploration 150 Verify that when A in Example 5.4.5 above is changed to the basis given for Jordan Canonical Form that the outcome is

$$J = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Note that in Example 5.4.5, we performed the same row reduction multiple times. This will always be the case if we compute our chains in this manner. If you'd prefer to avoid so many repetitive row reductions, you could instead find the product of elementary matrices that represents these row operations the first time you are doing them by augmenting with the identity matrix to keep track. Remember, this was one way we described how to find the inverse of a matrix back in Section 4.5. Then, you can do a simple matrix multiplication to see what the outcome would have been from the row reduction.

Example 5.4.6 Let's revisit Example 5.4.5 using the product of elementary matrices to see how those computations work. First, we augment $A-3I$ by the identity matrix and row reduce.

$$\left[\begin{array}{rrrrr|rrrrr} 2 & 2 & -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 2 & 1 & -3 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrr|rrrrr} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Notice that here we have lined up our pivots on the left hand side to be on the diagonal. This is then not in reduced row-echelon form, but this form will actually allow us to more quickly see our answers since we are always making the choice that the free variables are 0. Let us define now the matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

We find $\text{Ker } A-3I$ in the same way as before to get the same $\vec{v}_1$ and $\vec{v}_2$. Now, we can use B to find $\vec{v}_3$ and $\vec{v}_4$ since $B\vec{v}_1 = \vec{v}_3$ and $B\vec{v}_2 = \vec{v}_4$. Once we have $\vec{v}_3$ and $\vec{v}_4$, we can again use B to find $\vec{v}_6$ since $B\vec{v}_4 = \vec{v}_6$. To see that $\vec{v}_5$ does not exist, note that $B\vec{v}_3$ would have a nonzero entry for x_5, which does not align with our choice that $x_2 = x_5 = 0$.

You should all be prepared now for the exercises that await.

Section Highlights

- A matrix in *Jordan Canonical Form* has eigenvalues on the diagonal, 1's or 0's immediately above the diagonal, and 0's everywhere else. See Definition 5.4.2 and Theorem 5.4.1.

- While not all matrices in $\mathcal{M}_{n\times n}$ are diagonalizable, it is always possible to find a basis of $\mathbb{C}^n$ that puts a matrix into Jordan Canonical Form. See Theorem 5.4.1.

- Every matrix in $\mathcal{M}_{n\times n}$ has a unique Jordan Canonical Form (allowing for rearrangement of blocks), and two matrices are similar if and only if they have the same Jordan Canonical Forms (again, allowing for rearrangement of blocks). See Theorem 5.4.1 and Corollary 5.4.4.

- ► The procedure for computing Jordan Canonical Form is illustrated in Example 5.4.1 and Example 5.4.5.

Exercises for Section 5.4

5.4.1. Now you will create examples.

(a) Make a matrix in $\mathcal{M}_{4\times 4}$ with 4 Jordan blocks.

(b) Make a matrix in $\mathcal{M}_{4\times 4}$ with 3 Jordan blocks. Use those same Jordan blocks to make a different matrix in $\mathcal{M}_{4\times 4}$.

(c) Make a matrix in $\mathcal{M}_{4\times 4}$ with 2 Jordan blocks. Use those same Jordan blocks to make a different matrix in $\mathcal{M}_{4\times 4}$.

(d) Make a matrix in $\mathcal{M}_{4\times 4}$ with 1 Jordan block.

5.4.2. Find the Jordan Canonical Form for the following matrices.

(a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

5.4.3. Here are some matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 1 \\ 2 & -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 2 & 1 & 0 & -1 & -1 \\ -1 & 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

(a) A has eigenvalue 1 with algebraic multiplicity 3 and geometric multiplicity 1. Find a Jordan chain for 1 of length 3.

(b) B has eigenvalue 1 with algebraic multiplicity 5 and geometric multiplicity 1. Find a Jordan chain for 1 of length 5.

5.4.4. Let's explore the left shift! Define $L\colon \mathbb{R}^n \to \mathbb{R}^n$ by $L(\vec{x}_1, \ldots, \vec{x}_n) = (\vec{x}_2, \ldots, \vec{x}_n, 0)$.

(a) Prove that the left shift is a linear transformation.

(b) Find a matrix representation for L using the standard basis on $\mathbb{R}^n$.

(c) Find the kernel and image of L.

(d) Show that L is *nilpotent*; that is, show that there is some positive integer k such that $L^k = 0$, where 0 is the linear transformation that maps all vectors to $\vec{0}$.

5.4.5. Let $T\colon V \to V$ be a linear transformation with invariant subspaces $V_1, \ldots, V_n$ such that $V = V_1 \oplus \cdots \oplus V_n$. Show that T decomposes into the direct sum of linear transformations $T|_{V_1}, \ldots, T|_{V_n}$.

5.4.6. Complete the proof of Lemma 5.4.6.

5.4.7. Prove Lemma 5.4.11.

5.4.8. Prove Lemma 5.4.12.

5.5 Spectral Theory

As we saw in Section 5.3, some square matrices are diagonalizable, which is neat, but as we just saw in Section 5.4, *all* matrices are very nearly diagonalizable. Great, but you probably still wonder what precisely it is about a matrix that makes it diagonalizable. Sure, the diagonalizability of an $n \times n$ matrix is characterized by having n linearly independent eigenvectors, but what's that all about? Why should some matrices have a maximal number of invariant subspaces and others not? That is a good question, and we shall see that it has something like a good answer.

Symmetric Matrices

Recall that A^T, the transpose of the matrix A, is a matrix whose columns are the rows of A.[41]

41: This is true for all kinds of matrices, real or complex.

What about matrices of abstract shapes?

The definition still works, but what are you doing with abstract shapes?

You don't understand my art!

Definition 5.5.1 *A* ***symmetric*** *matrix is a matrix A such that $A^T = A$.*

Symmetric matrices must be square, and entries "mirror each other across the main diagonal."

Example 5.5.1 Here are some matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Note that $A^T = A$ (check it!), so A is symmetric. You could also note that $a_{12} = a_{21} = 2$, $a_{13} = a_{31} = 3$, and $a_{23} = a_{32} = 5$. However, B is not symmetric because $b_{12} = 2 \neq 3 = b_{21}$.

These neatly organized matrices we call "symmetric" have extremely nice properties. You'll recall from Theorem 5.1.2 that we already know eigenvectors from different eigenspaces are always going to be linearly independent. The additional hypothesis of symmetry improves this linear independence to orthogonality.

Theorem 5.5.1 *If a matrix A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.*

PROOF OF THEOREM 5.5.1. First of all, recall from Theorem 4.4.9 that for any two appropriately sized matrices B and C, we know $(BC)^T = C^T B^T$. Suppose $A\vec{v}_1 = \lambda_1 \vec{v}_2$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$, where $\lambda_1 \neq \lambda_2$. Since A is symmetric, we know $A^T = A$. Then

$$\begin{aligned} \lambda_1 \vec{v}_1 \cdot \vec{v}_2 &= A\vec{v}_1 \cdot \vec{v}_2 \\ &= (A\vec{v}_1)^T \vec{v}_2 \\ &= \vec{v}_1^T A^T \vec{v}_2 \\ &= \vec{v}_1^T A \vec{v}_2 \\ &= \vec{v}_1^T \lambda_2 \vec{v}_2 \;=\; \lambda_2 \vec{v}_1 \cdot \vec{v}_2. \end{aligned}$$

Then $(\lambda_1-\lambda_2)(\vec{v}_1\cdot\vec{v}_2)=0$. Since $\lambda_1-\lambda_2\neq 0$, we must have $\vec{v}_1\cdot\vec{v}_2=0$. $\square$

Example 5.5.2 Let's try Theorem 5.5.1 on the symmetric matrix

$$A=\begin{bmatrix} 1 & -6 & 4\\ -6 & 2 & -2\\ 4 & -2 & -3\end{bmatrix}$$

One can check that the eigenvalues and eigenvectors are as follows:

$$\lambda_1=9,\qquad \vec{v}_1=\begin{bmatrix} 2\\ -2\\ 1\end{bmatrix},$$

$$\lambda_2=-6,\qquad \vec{v}_2=\begin{bmatrix} -2\\ -1\\ 2\end{bmatrix},\quad\text{and}$$

$$\lambda_3=-3,\qquad \vec{v}_3=\begin{bmatrix} 1\\ 2\\ 2\end{bmatrix}$$

One can also verify that $\vec{v}_1\cdot\vec{v}_2=\vec{v}_1\cdot\vec{v}_3=\vec{v}_2\cdot\vec{v}_3=0$.

Exploration 151 Find the eigenvalues and eigenvectors for the symmetric matrix

$$A=\begin{bmatrix} 3 & 4\\ 4 & -3\end{bmatrix}.$$

Verify that the eigenvectors are orthogonal.

The Spectral Theorem

Let's start with the most tongue-twistery terminology.

Definition 5.5.2 *A matrix A is **orthogonally diagonalizable** if there is an orthogonal matrix P and a diagonal matrix D such that*

$$A=PDP^T=PDP^{-1}.$$

Example 5.5.3 Orthogonally diagonalize

$$A=\begin{bmatrix} 1 & -6 & 4\\ -6 & 2 & -2\\ 4 & -2 & -3\end{bmatrix}.$$

We know from Example 5.5.2 that

$$\left\{\begin{bmatrix} 2/3\\ -2/3\\ 1/3\end{bmatrix},\ \begin{bmatrix} -2/3\\ -1/3\\ 2/3\end{bmatrix},\ \begin{bmatrix} 1/3\\ 2/3\\ 2/3\end{bmatrix}\right\}$$

is an orthogonal set of eigenvectors. Observe that we have also normalized each of them, so this is actually an ortho*normal* set of eigenvectors. If we define,

$$P = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

then we can then check that $A = PDP^T$.

Theorem 5.5.2 *If a matrix A is orthogonally diagonalizable, then A is symmetric.*

PROOF. Suppose $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix. Then

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$$

□

Theorem 5.5.2 is only half the story. It's actually true that A is orthogonally diagonalizable *if and only if* A is symmetric. The following theorem, known as the Spectral Theorem,[42] makes it so; we leave the proof of this theorem for the end of the section.

42: The *spectrum* of a matrix A is the set of all eigenvalues, and the *spectral radius* is the maximum magnitude of all the eigenvalues of A.

Yeah, that's not nearly as cool as you probably thought it would be. Sorry.

Theorem 5.5.3 (The Real Spectral Theorem) *A symmetric matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the following properties:*

(a) *A has n real eigenvalues, counting multiplicities.*
(b) *For each eigenvalue λ, the geometric multiplicity of λ equals the algebraic multiplicity of λ.*
(c) *The eigenspaces of A are mutually orthogonal.*
(d) *A is orthogonally diagonalizable.*

Besides being amazing, the Spectral Theorem is also quite useful. Behold!

Definition 5.5.3 *The **spectral decomposition** of an orthogonally diagonalizable matrix A is*

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T,$$

where for $1 \leq i \leq n$, λ_i are eigenvalues of A, and $\vec{u}_i$ are corresponding orthonormal eigenvectors of A.

Note that each $\vec{u}_i \vec{u}_i^T$ is a matrix that projects onto the subspace spanned by $\vec{u}_i$. In fact, we can make[43] a more general statement about such projection matrices.

43: And prove!

Theorem 5.5.4 *Let $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$, and let $A = [\vec{w}_1 \cdots \vec{w}_k]$. For all $\vec{v} \in \mathbb{R}^n$,*

$$\text{proj}_W(\vec{x}) = AA^T\vec{x}.$$

PROOF. Recall from way back in Chapter 2 that

$$\text{proj}_W(\vec{x}) = \frac{\vec{x}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \cdots + \frac{\vec{x}\cdot\vec{w}_k}{\vec{w}_k\cdot\vec{w}_k}\vec{w}_k.$$

Using the fact that vectors in $\mathcal{B}$ are unit vectors, the fact that the inner product is commutative, and rewriting our inner products as matrix products, we have

$$\begin{aligned}\operatorname{proj}_W(\vec{x}) &= (\vec{x}\cdot\vec{w}_1)\vec{w}_1 + \cdots + (\vec{x}\cdot\vec{w}_k)\vec{w}_k \\ &= (\vec{w}_1\cdot\vec{x})\vec{w}_1 + \cdots + (\vec{w}_k\cdot\vec{x})\vec{w}_k \\ &= (\vec{w}_1^T\vec{x})\vec{w}_1 + \cdots + (\vec{w}_k^T\vec{x})\vec{w}_k\end{aligned}$$

Then we have

$$\operatorname{proj}_W(\vec{x}) = [\vec{w}_1 \cdots \vec{w}_k]\begin{bmatrix}\vec{w}_1^T\vec{x}\\ \vdots \\ \vec{w}_k^T\vec{x}\end{bmatrix} = AA^T\vec{x}.$$

□

Example 5.5.4 Let's find the spectral decomposition of

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}.$$

Note first that A is symmetric, so we know from the Spectral Theorem that the spectral decomposition exists. Here are the eigenvalues and eigenvectors of A:

$$\lambda_1 = 10, \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \lambda_2 = 5, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We need unit vectors, so let's use

$$\vec{u}_1 = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix}.$$

Then

$$\begin{aligned}\lambda_1\vec{u}_1\vec{u}_1^T + \lambda_2\vec{u}_2\vec{u}_2^T &= 10\begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}\begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \end{bmatrix} \\ &\quad +5\begin{bmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix}\begin{bmatrix} 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix} \\ &= 10\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} + 5\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}\end{aligned}$$

This is the spectral decomposition of A. When written in this fashion, A is a linear combination of projections, which is computationally efficient. It is easy to verify from this point that this linear combination is equal to A:

$$\begin{aligned}10\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} + 5\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} &= \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}\end{aligned}$$

The Complex Spectral Theorem

Definition 5.5.4 *For $z = x + iy \in \mathbb{C}$, the **conjugate of z** is the complex number $\bar{z} = x - iy$. For $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, the **conjugate of A** is the matrix $\overline{A} = [\bar{a}_{ij}] \in \mathcal{M}_{m\times n}(\mathbb{C})$ obtained by conjugating every entry A. The*

***conjugate transpose of** A, denoted by A^H, is obtained by conjugating the transpose of A; that is, $A^H = \overline{A^T}$.*

Definition 5.5.5 *A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is called **Hermitian** if $A = A^H$.*

Example 5.5.5 Both of these matrices are Hermitian, but only one is symmetric.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2+3i & 4 \\ 2-3i & 5 & 7-8i \\ 4 & 7+8i & 9 \end{bmatrix}$$

Exploration 152 Prove that all symmetric matrices in $\mathcal{M}_{n\times n}(\mathbb{R})$ are Hermitian. Also, find an example to show that there are matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$ that are symmetric but not Hermitian.

Exploration 153 Prove that all Hermitian matrices have real entries on their main diagonal.

There is a complex analog of Theorem 5.5.1 for Hermitian matrices. Its proof is nearly identical to the proof of Theorem 5.5.1. However, just as in Section 5.4, we need to use the Hermitian inner product and norm (Definitions 5.4.7 and 5.4.8) when dealing with matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$. Other than *that*, the proof is nearly identical.

Corollary 5.5.5 *If A is Hermitian, then any two eigenvectors from different eigenspaces are orthogonal.*

Just like how Hermitian matrices are the complex generalization of symmetric matrices, we have a complex generalization of orthogonal matrices.

Definition 5.5.6 *A matrix $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ is called **unitary** if $UU^H = U^HU = I_n$.*

Unitary matrices enjoy many properties analogous to the nice properties enjoyed by real symmetric matrices:

Theorem 5.5.6 *Let $U \in \mathcal{M}_{n\times n}(\mathbb{C})$. The following conditions are equivalent:*

(a) *U is unitary.*
(b) *U is invertible with $U^{-1} = U^H$.*

(c) *The columns of U are an orthonormal basis for $\mathbb{C}^n$.*

We leave the proof of Theorem 5.5.6 as an exercise.

Here's a surprising fact. Well, it's surprising if you're not at all familiar with the Spectral Theorem.

Theorem 5.5.7 *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is Hermitian, then all of the eigenvalues of A are real.*

PROOF. We know from the Fundamental Theorem of Algebra that $\det(A - \lambda I) = 0$ has a solution, which by definition is an eigenvalue. Suppose $\lambda = x + iy$ for some $x, y \in \mathbb{R}$ and that $\vec{v}$ is an associated eigenvector, so $A\vec{v} = \lambda\vec{v}$. We will make extensive use of our Hermitian inner product and use the fact that for any $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$ we have $(AB)^H = B^H A^H$ and $A = (A^H)^H$.[44] Note first that

$$\begin{aligned}
(A\vec{v})\cdot\vec{v} &= \vec{v}^H(A\vec{v}) && \text{by conversion to matrix multiplication,}\\
&= (\vec{v}^H A)\vec{v} && \text{by associativity of matrix multiplication,}\\
&= (A^H\vec{v})^H\vec{v} && \text{since } (AB)^H = B^H A^H \text{ and } A = (A^H)^H,\\
&= \vec{v}\cdot(A^H\vec{v}) && \text{by conversion back to inner product,}\\
&= \vec{v}\cdot(A\vec{v}) && \text{since } A \text{ is Hermitian,}\\
&= \vec{v}\cdot(\lambda\vec{v}) &&\\
&= \bar{\lambda}(\vec{v}\cdot\vec{v}) && \text{by definition of Hermitian inner product.}
\end{aligned}$$

Also, we have

$$\begin{aligned}
(A\vec{v})\cdot\vec{v} &= (\lambda\vec{v})\cdot\vec{v} &&\\
&= \lambda(\vec{v}\cdot\vec{v}) && \text{by definition of Hermitian inner product.}
\end{aligned}$$

It follows that $\bar{\lambda}(\vec{v}\cdot\vec{v}) = \lambda(\vec{v}\cdot\vec{v})$, but since $\vec{v}$ is an eigenvector, we know that $\vec{v}\cdot\vec{v} > 0$. Thus, $\bar{\lambda} = \lambda$, or $x + iy = x - iy$. This implies that $y = -y$, so $y = 0$. Then $\lambda = x \in \mathbb{R}$. □

44: Exercise!

Theorem 5.5.8 (The Spectral Theorem) *If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is Hermitian, then there is a unitary matrix U and a real diagonal matrix D such that $A = U^H D U$.*

The proof is similar in flavor to the proof of Theorem 5.4.7, but the fact that A is Hermitian will make our calculations much nicer. Let's have a lemma first.

Lemma 5.5.9 *For square matrices A and B, $(AB)^H = B^H A^H$.*

PROOF. We will use the fact that for $z, w \in \mathbb{C}$, $\overline{zw} = \bar{z}\bar{w}$, and you will prove that as an exercise.

$$(AB)^H = \overline{(AB)^T} = \overline{B^T A^T} = B^H A^H.$$

□

PROOF OF THEOREM 5.5.8. Let λ_1 be an eigenvalue for A; we know λ_1 exists from the Fundamental Theorem of Algebra, and we know that λ_1 is real from Theorem 5.5.7. Let $\vec{x}_1$ be one of its associated eigenvectors such that $\|\vec{x}_1\| = 1$ and $E_1 = \text{Span}\{\vec{x}_1\}$. Then $\dim E_1^\perp = n - 1$. Make a basis for

$\mathbb{C}^n$ using $\vec{x}_1$ and $n-1$ orthonormal vectors, $\vec{v}_2, \ldots, \vec{v}_n \in E_1^{\perp}$, and define $Q_1 = [\vec{x}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$. By Theorem 5.5.6, Q_1 is a unitary matrix, and

$$Q_1^H A Q_1 = \begin{bmatrix} \vec{x}_1^H \\ \vec{v}_2^H \\ \vdots \\ \vec{v}_n^H \end{bmatrix} [\lambda_1 \vec{x}_1\ A\vec{v}_2 \cdots A\vec{v}_n].$$

Observe that $\vec{x}_1^H \vec{x}_1 = \|\vec{x}_1\| = 1$ and $\vec{v}_i^H \vec{x}_1 = 0$ for $i = 2, \ldots, n$, so the first column of $Q_1^H A Q_1$ is $\lambda_1 \vec{e}_1$. Moreover, since

$$(Q_1^H A Q_1)^H = Q_1^H A^H (Q_1^H)^H = Q_1^H A Q_1,$$

$Q_1^H A Q_1$ is Hermitian. Thus,

$$Q_1^H A_1 Q_1 = \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & C_1 \end{bmatrix},$$

where $\vec{0} \in \mathcal{M}_{n-1,1}$ and $C_1 \in \mathcal{M}_{n-1,n-1}(\mathbb{C})$ is also Hermitian. Since C_1 is Hermitian, this is a very nice and repeatable procedure.

Again, C_1 has a real eigenvalue, call it λ_2, with an eigenvector, call it $\vec{x}_2$, such that $\|\vec{x}_2\| = 1$ and $E_2 = \operatorname{Span}\{\vec{x}_2\}$. We can build a unitary matrix using $\vec{e}_1$, $\vec{x}_2$, and $n-2$ more orthonormal vectors, $\vec{u}_2, \ldots, \vec{u}_n \in E_2^{\perp}$ and define $Q_2 = [\vec{e}_1\ \vec{x}_2\ \vec{u}_2\ \cdots\ \vec{u}_n]$. Then

$$Q_2^H Q_1^H A Q_1 Q_2 = Q_2^H \begin{bmatrix} \lambda_1 & \vec{0}^T \\ \vec{0} & C_1 \end{bmatrix} Q_2 = \begin{bmatrix} \lambda_1 & 0 & \vec{0}^T \\ 0 & \lambda_2 & \vec{0}^T \\ \vec{0} & \vec{0} & C_2 \end{bmatrix},$$

where $\vec{0} \in \mathcal{M}_{n-2,1}$ and $C_2 \in \mathcal{M}_{n-2,n-2}(\mathbb{C})$ is also Hermitian.

Do this $n-2$ more times, and define $U = Q_1 Q_2 \ldots Q_{n-1}$. It remains only to verify that Q is unitary.[45] □

45:  As an exercise!

The proof of Theorem 5.5.3 now follows quickly, but for reference, we provide the details.

PROOF OF THEOREM 5.5.3. Part a follows from Theorem 5.5.7, part b follows from part a, and part c follows from Theorem 5.5.1]. Since $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is symmetric, it is Hermitian, so part d follows from Theorem 5.5.8, noting that since A is had only real entries, U is a symmetric matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$ by construction. □

Section Highlights

- ▶ A matrix is orthogonally diagonalizable if there exists an orthogonal basis with respect to which it is diagonal. See Definition 5.5.2.
- ▶ A matrix is called symmetric if it is equal to its own transpose. See Definition 5.5.1.
- ▶ A real-valued matrix will be orthogonally diagonalizable if and only if it is symmetric. Thus, a symmetric real-valued matrix is always

diagonalizable and always has all real eigenvalues. This is part of the Real Spectral Theorem. See Theorem 5.5.2 and Theorem 5.5.3.

- For any real-valued symmetric matrix A, there is a spectral decomposition that decomposes A into a sum of symmetric matrices scaled by the eigenvalues of A. See Definition 5.5.3 and Example 5.5.4.
- For matrices with complex entries, the concept of Hermitian matrices replaces that of symmetric matrices, and there is a more-general version of the Spectral Theorem. See Definition 5.5.5, Theorem 5.5.7, Definition 5.5.6, and Theorem 5.5.8.

Exercises for Section 5.5

5.5.1. Here's a symmetric matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

According to Theorem 5.5.1, any two eigenvectors from different eigenspaces are orthogonal. Verify this is true for A.

5.5.2. Let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Show that PDP^{-1} is not symmetric.

(b) Use the Gram-Schmidt process on the columns of P, normalize the resulting three vectors, and use them to make an orthogonal matrix Q.

(c) Show that QDQ^{-1} *is* symmetric.

5.5.3. Find the spectral decomposition for each of the following matrices.

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

5.5.4. Prove that for any $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$, $(AB)^H = B^H A^H$. *Hint: See Theorem 4.4.9 and make sure to keep track of the conjugates from the Hermitian inner product.*

5.5.5. Prove Theorem 5.5.6.

5.5.6. Recall that for $z = x + iy \in \mathbb{C}$, $\bar{z} = x - iy$. Prove that for $z, w \in \mathbb{C}$, $\overline{zw} = \bar{z}\bar{w}$.

5.5.7. Prove that if $A_1, \ldots, A_n \in \mathcal{M}_{n\times n}(\mathbb{C})$ are unitary, then $A_1 \cdots A_n$ is unitary.

5.6 Singular Value Decomposition

We've spent the entire chapter to this point dealing only with square matrices. Now we'll take what we learned and apply it to the more general setting of rectangular matrices. No mucking about. Let's have the theorem.

Theorem 5.6.1 (Real Singular Value Decomposition) *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$. Then there is an orthogonal matrix $U \in \mathcal{M}_{m\times m}(\mathbb{R})$, an orthogonal matrix $V \in \mathcal{M}_{n\times n}(\mathbb{R})$, and a rectangular diagonal matrix $D \in \mathcal{M}_{m\times n}(\mathbb{R})$ such that*

$$A = UDV^T.$$

It doesn't take a lot of imagination to correctly guess what a *rectangular diagonal* matrix is. We're not giving the formal definition until you guess.

...ok. Ready?

Definition 5.6.1 *A matrix $D \in \mathcal{M}_{m\times n}$ with entries $d_{i,j}$ is called a* ***rectangular diagonal matrix*** *if $d_{i,j} = 0$ whenever $i \neq j$.*

For example, matrices of the form

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{mm} & 0 & \cdots & 0 \end{bmatrix}$$

or

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

are rectangular diagonal matrices.

Example 5.6.1 Here's a non-square matrix:

$$A = \begin{bmatrix} -20 & 8 & -2 \\ -10 & 19 & 14 \end{bmatrix}$$

You can check that $A = UDV^T$, $UU^T = U^TU = I_3$, and $VV^T = V^TV = I_4$, where

$$\begin{aligned} U &= \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}, \\ D &= \begin{bmatrix} 15 & 0 & 0 \\ 0 & 30 & 0 \end{bmatrix}, \text{ and} \\ V^T &= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}. \end{aligned}$$

How do you actually find such a decomposition? We need one more definition.

Definition 5.6.2 *A nonnegative $\sigma \in \mathbb{R}$ is a* ***singular value*** *for $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ if there are unit vectors $\vec{u} \in \mathbb{R}^m$ and $\vec{v}\mathbb{R}^n$ such that*

$$A\vec{v} = \sigma\vec{u} \qquad \text{and} \qquad A^T\vec{u} = \sigma\vec{v}.$$

In this case, $\vec{u}$ and $\vec{v}$ are called ***left-singular*** *and* ***right-singular*** *vectors, respectively.*

The left-singular vectors will be the columns of U, the right-singular vectors will be the columns of V, and the singular values will be the diagonal entries on D. If you're willing to believe that the singular value decomposition exists (and you should; we'll prove it soon), then finding the singular values and vectors is not terribly difficult. Note first that since U and V are orthogonal and D is rectangular diagonal, we have

$$\begin{aligned} AA^T &= (UDV^T)(UDV^T)^T = (UDV^T)(VD^TU^T) = U(DD^T)U^T \\ A^TA &= (UDV^T)^T(UDV^T) = (VD^TU^T)(UDV^T) = V(D^TD)V^T \end{aligned}$$

Here are some handy facts that we'll prove later:

Theorem 5.6.2 *If $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, then AA^T and A^TA are both symmetric.*

Corollary 5.6.3 *If $D \in \mathcal{M}_{m\times n}(\mathbb{R})$ is rectangular diagonal, then DD^T and D^TD are both diagonal and have the same nonzero entries.*

Since AA^T and A^TA are both symmetric, we know from the Spectral Theorem that the columns of U and V should normalized eigenvectors of AA^T and A^TA, respectively. It's a little less obvious, but the eigenvalues of A, which become the entries on the diagonal of D, are *the square roots* of the eigenvalues of AA^T (or A^TA).

The only catch is that normalized eigenvectors aren't necessarily left-singular or right-singular vectors, but they do give us a starting point. If $\vec{u}$ is an eigenvector for AA^T, then calculate $A^T\vec{u}$. It should be one of the eigenvectors for A^TA. One may need to rescale some of these vectors by -1, but in this fashion, one can find and properly order the left-singular and right-singular vectors.

Example 5.6.2 Here's a matrix A with AA^T and A^TA:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

As indicated by Theorem 5.6.2, both AA^T and A^TA are symmetric. Here is some additional handy information:

Eigenvalues/eigenvectors for AA^T

eigenvalue	eigenvectors
$\sqrt{3}$	$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$
1	$\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

Eigenvalues/eigenvectors for A^TA

eigenvalue	eigenvectors
$\sqrt{3}$	$\vec{v}_1 = \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
1	$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
0	$\vec{v}_4 = \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Using the eigenvalues for AA^T, we have singular values 1, 1, and $\sqrt{3}$, so

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

You can check that $A\vec{v}_1 = \sqrt{3}\vec{u}_1$. However, $A\vec{v}_2 = -\vec{u}_3$ and $A^T\vec{u}_2 = -\vec{v}_3$. Thus, $\vec{u}_1$ and $\vec{v}_1$ are left-singular and right-singular vectors, but we need to use $-\vec{u}_3$ and $-\vec{u}_2$ (in that order) to have left-singular vectors if we use $\vec{v}_2$ and $\vec{v}_3$ as left-singular vectors. Defining

$$U = [\vec{u}_1 \ -\vec{u}_3 \ -\vec{u}_2] \qquad \text{and} \qquad V = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4],$$

we have the singular value decomposition $A = UDV^T$.

Exploration 154 Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The proof of Theorem 5.6.1 uses The Spectral Theorem and follows the roughly same procedure from Example 5.6.2, so we'll just take care of it in the exercises.

Pseudoinverses

We come immediately to a convenient application of the singular value decomposition. We also get a fun new term.

Definition 5.6.3 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ have singular value decomposition UDV^T, r be $\min\{m,n\}$, k be the number of nonzero singular values, and $\sigma_1,\ldots,\sigma_k,0_{k_1},\ldots,0_r$ be the diagonal entries of D. The* ***pseudoinverse*** *of A, denoted A^+, is the matrix VD^+U^T, where $D^+ \in \mathcal{M}_{n\times m}$ is rectangular diagonal with diagonal entries $1/\sigma_1,\ldots,1/\sigma_k,0,\ldots,0$.*

Example 5.6.3 Here's a matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Its singular value decomposition is UDV^T, where

$$\begin{aligned} U &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ D &= \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and} \\ V &= \begin{bmatrix} 2/\sqrt{6} & 0 & 0 & -1\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}. \end{aligned}$$

The pseudoinverse of A is $A^+ = VD^+U^T$, where

$$D^+ = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have

$$A^+ = VD^+U^T = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}.$$

That's a neat thing to do. What can we do this this fun new matrix, though? Recall that if A is invertible, A^{-1} can be used to find solutions to the matrix equation $A\vec{x} = \vec{b}$ by multiplying both sides of the equation by A^{-1}, so we have $\vec{x} = A^{-1}\vec{b}$. The compelling name, *pseudoinverse*, strongly suggests we can find vectors that are *almost* solutions; no invertibility required! That sure sounds familiar…

Theorem 5.6.4 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$. Then $AA^+\vec{b} = \operatorname{proj}_{\text{Col } A}\left(\vec{b}\right)$. In particular, $\hat{x}$ is a least squares solution to $A\vec{x} = \vec{b}$ if and only if $\hat{A}x = AA^+\vec{b}$.*

PROOF. Recall that $\hat{x}$ is a least squares solution if and only if it holds that $A\hat{x} = \text{proj}_{\text{Col }A}\left(\vec{b}\right)$. Note that $\hat{x} = A^+\vec{b}$ if and only if $A\hat{x} = AA^+\vec{b}$, or

$$A\hat{x} = (UDV^T)(VD^+U^T)\vec{b} = UU^T\vec{b}.$$

Recall from Theorem 5.5.4 that $UU^T\vec{b} = \text{proj}_{\text{Col }U}\left(\vec{b}\right)$. Since you will prove Col $A =$ Col U as an exercise, we are done. $\square$

There is a subtle advantage here. To find the least squares solutions to $A\vec{x} = \vec{b}$, where $A \in \mathcal{M}_{m\times n}$, we can solve either

$$A^TA\vec{x} = A^T\vec{b}, \text{ or} \tag{5.2}$$

$$A\vec{x} = AA^+\vec{b}. \tag{5.3}$$

The augmented matrix for Equation 5.2 is n by $n+1$. The augmented matrix for Equation 5.6.3 is still m by $n+1$, just like the original augmented matrix..

Example 5.6.4 Using

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$A^+ = \frac{1}{8}\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & -1 & 0 \end{bmatrix}.$$

One can quickly check that $\vec{e}_1 + \vec{e}_3 \notin$ Col A. To find the least squares solutions for $A\vec{x} = \vec{e}_1 + \vec{e}_3$ using the normal equation $A^TA\vec{x} = A^T(\vec{e}_1 + \vec{e}_3)$, we have a 6×7 auxiliary matrix. However, note that the auxiliary matrix for $A\vec{x} = AA^+\vec{b}$ is 3×7, which is substantially smaller and easier to solve. One can check that $AA^+\vec{b} = \vec{e}_1$ and that the augmented matrix for $A\vec{x} = AA^+(\vec{e}_1 + \vec{e}_3)$ is

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so the least squares solutions for $A\vec{x} = \vec{e}_1 + \vec{e}_3$ are $\hat{x} =$

$$x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $x_2, x_4, x_5, x_6, x_7 \in \mathbb{R}$.

Great. The pseudoinverse does nice things for us, but it's also the *unique* matrix that does so... in the following very specific sense. This nicely mirrors the fact that actual inverses of invertible matrices are unique.

Theorem 5.6.5 *For $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$ singular value decomposition UDV^T, the pseudoinverse $A^+ = VD^+U^T$ is the unique matrix satisfying all of the following properties:*

(a) $AA^+A = A$
(AA^+ maps the columns of A to the columns of A),

(b) $A^+AA^+ = A^+$
(A^+A maps the columns of A^+ to the columns of A^+),

(c) $(AA^+)^T = AA^+$
(AA^+ is symmetric), and

(d) $(A^+A)^T = A^+A$
(A^+A is symmetric).

Proof of Theorem 5.6.5 can be obtained by direct calculation, so... exercise!

The Complex Singular Value Decomposition

Corollary 5.6.6 *If $A \in \mathcal{M}_{n\times m}(\mathbb{C})$, then $AA^H \in \mathcal{M}_{n\times n}(\mathbb{R})$, $A^HA \in \mathcal{M}_{m\times m}(\mathbb{R})$, and both AA^H and A^HA are Hermitian.*

This corollary comes from clever conjugation. Exercise! Proof of the following theorem then follows from the (complex) Spectral Theorem.

Theorem 5.6.7 (Singular Value Decomposition) *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Then there is a unitary matrix $U \in \mathcal{M}_{m\times m}(\mathbb{C})$, a unitary matrix $V \in \mathcal{M}_{n\times n}(\mathbb{C})$, and a rectangular diagonal matrix $D \in \mathcal{M}_{m\times n}(\mathbb{R})$ such that*

$$A = UDV^H.$$

It's worth noting that this works even if A has complex entries, and in that case, D still has only real entries.

Exploration 155 Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ i & i & i & i & i \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Section Highlights

- While diagonalization was a topic for square matrices, there is a way to decompose any matrix using the *singular value decomposition*. See Theorem 5.6.1 and Example 5.6.2.
- One application of the singular value decomposition is the existence of a pseudoinverse, which can be used to compute least squares solutions. See Definition 5.6.3 and Theorem 5.6.4.

Exercises for Section 5.6

5.6.1. We will prove Theorem 5.6.1 in a few steps. Suppose $n > m$. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$. Use the Spectral Theorem on $A^TA \in \mathcal{M}_{n\times n}(\mathbb{R})$ to get orthonormal vectors $\vec{v}_1, \ldots, \vec{v}_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicity). For each $i = 1, \ldots, n$, define $\sigma_i = \sqrt{\lambda_i} > 0$.

(a) Use the σ_is to define D.

(b) For each $i = 1, \ldots, n$, define

$$\vec{u}_i = \frac{1}{\sigma_i}A\vec{v}_i,$$

and verify that $U = [\vec{u}_1 \cdots \vec{u}_n]$ is orthogonal.

(c) Verify that $AV = UD$ and use that to show $A = UDV^T$.

(d) Now suppose $m > n$, and verify that a similar proof works after using the Spectral Theorem on AA^T.

(e) Complete the proof by verifying the $m = n$ case.

5.6.2. Prove Corollary 5.6.6.

5.6.3. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$ have singular value decomposition UDV^T. Verify that $\text{Col } A = \text{Col } U$.

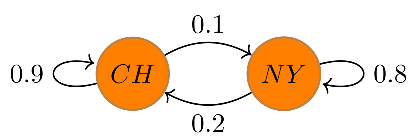
5.6.4. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R}^n)$ have singular value decomposition UDV^T. Prove Theorem 5.6.5.

5.7 Applications of Invariant Subspaces

We've constructed an impressive catalog of incredible theorems, and there are dozens of practical uses for each of them. We will show you two.

Discrete Dynamics and Markov Chains

The fictitious city of Narwhal Springs has exactly two restaurants, and they both serve pizza exclusively. All is not well in Narwhal Springs, though. One restaurant serves Chicago deep-dish pizza, the other serves New York thin-crust pizza, and it turns out some people have very strong feelings about these two types of pizza. A weekly survey of the townspeople is conducted in which they are forced to choose which type of pizza they prefer. People, being very entrenched in their pizza preferences, tend to stick with the same type of pizza they currently prefer. Specifically, next week 90% of people that prefer Chicago pizza will still prefer Chicago pizza, and 80% of people that prefer New York pizza will still prefer New York pizza. That means 10% of Chicago pizza people convert to New York pizza people, and 20% of New York pizza people convert to Chicago pizza. Here is a convenient diagram, called a *transition diagram*, that summarizes all of this information:



Here are some important features of this situation:

- The probabilities predicting future preference only depend on the current preference.
- There are only a finite number of choices (two in this case).

We can put this situation into action using vectors and matrices. First let's start with a vector representing the distribution of people's pizza preference:

$$\vec{x} = \begin{bmatrix} c \\ n \end{bmatrix} \begin{array}{l} \longleftarrow \text{proportion of people preferring Chicago pizza} \\ \longleftarrow \text{proportion of people preferring New York pizza} \end{array}$$

Note that the entries of $\vec{x}$ must sum to 1; that is $c + n = 1$. This is because these are the proportions,[46] and we have proportions associated to all possible outcomes; in this case, there are only two possible restaurant choices, and we made people pick one. Using our transition diagram, we can predict the next week's proportions of pizza preference,

46:  We could also think about them as probabilities.

$$\begin{bmatrix} 0.9c + 0.2n \\ 0.1c + 0.8n \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} c \\ n \end{bmatrix} = A\vec{x},$$

which we can also write as a matrix, A, multiplied by $\vec{x}$. Such a matrix is sometimes called a *transition matrix*, and we can use $A\vec{x}$, $A^2\vec{x}$, $A^3\vec{x}$, and so on to predict how weekly changes in pizza preference progress.

Let's get more formal:

Definition 5.7.1 *Given a finite set of* ***states****, $\{1, 2, \ldots, n\}$, in which the probability of transition from the current state to another depends only on the current state, a* ***Markov chain*** *is a sequence describing how a distribution amongst the states evolves as a result of these probabilities.*

A common way to represent Markov chains is with vectors and matrices:

Definition 5.7.2 *A vector whose entries are all nonnegative and sum to 1 is called a* ***probability vector****. A square matrix whose columns are all probability vectors is called a* ***transition matrix****.*

Theorem 5.7.1 *The product of a transition matrix and a probability vector is a probability vector.*

Exploration 156 Let's prove Theorem 5.7.1 for a 2×2 transition matrix and a probability vector in $\mathbb{R}^2$. Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a transition matrix. Then $a + c = 1$ and $b + d = 1$. Suppose

$$\vec{x} = \begin{bmatrix} e \\ f \end{bmatrix}$$

is a probability vector. Then $e + f = 1$. Compute $A\vec{x}$ and show it is also a probability vector.

If A is a transition matrix, and $\vec{x}$ is a probability vector, then $\vec{x}, A\vec{x}, A^2\vec{x}, A^3\vec{x}, \ldots$ is a Markov chain. Transition matrices also have another convenient property.

Theorem 5.7.2 *If $A \in \mathcal{M}_{n\times n}$ is a transition matrix with eigenvalue λ, then $|\lambda| \leq 1$. Moreover, A has 1 as an eigenvalue.*

PROOF. First, suppose λ is an eigenvalue for A with $|\lambda| > 1$ with eigenvector $\vec{x}$. For large enough k, we have that $\|A^k\vec{x}\| = \|\lambda^k\vec{x}\| > (|x_1|+\cdots+|x_n|)$. By Theorem 5.7.1 and the definition of matrix multiplication, we know that A^k is a transition matrix, so each column of A^k, $\vec{a}_i$ for $i = 1, \ldots, n$, has $\|\vec{a}_i\| \leq 1$. Observe that

$$\begin{aligned} \|A^k\vec{x}\| &= \|x_1\vec{a}_1 + \cdots + x_n\vec{a}_n\| \\ &\leq |x_1|\|\vec{a}_1\| + \cdots + |x_n|\|\vec{a}_n\| \\ &\leq |x_1| + \cdots + |x_n|. \end{aligned}$$

However, we already have that $\|A^k\vec{x}\| > (|x_1|+\cdots+|x_n|)$, so it must be that there are no eigenvalues λ with $|\lambda| > 1$.

To see that A has an eigenvalue of 1, note first that since the entries in every column of A sum to 1, we have

$$A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

so 1 is an eigenvalue for A^T. By Theorem 5.3.3, A and A^T have the same characteristic polynomial, so 1 is an eigenvalue for A as well. $\square$

Example 5.7.1 Recall the preceding pizza problem had transition matrix

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

Then starting with two different probability vectors, $\vec{x}$ and $\vec{y}$,

$$\begin{array}{rclrcl}
\vec{x} & = & \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} & \vec{y} & = & \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} \\
A\vec{x} & = & \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} & A\vec{y} & = & \begin{bmatrix} 0.27 \\ 0.73 \end{bmatrix} \\
A^2\vec{x} & = & \begin{bmatrix} 0.585 \\ 0.415 \end{bmatrix} & A^2\vec{y} & = & \begin{bmatrix} 0.389 \\ 0.611 \end{bmatrix} \\
& \vdots & & & \vdots & \\
A^{100}\vec{x} & \approx & \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} & A^{100}\vec{y} & \approx & \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}
\end{array}$$

are both Markov chains. It's not surprising, based on the transition diagram, that the proportion of Chicago pizza preferences increases and New York pizza preferences decreases each week. Perhaps what is surprising is that it seems to settle, in both cases, to a specific set of proportions. There is actually an eigenvalue-based reason that this is happening; you can check that the eigenvalues for A are 1 and 0.7, with eigenvectors,

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

respectively. Since $\{\vec{v}_1, \vec{v}_2\}$ is a basis, any vector can be written as $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some scalars c_1 and c_2. Then

$$\begin{aligned}
A^k\vec{x} &= A\left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \\
&= c_1 A^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 A^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= c_1 (1)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\lim_{k\to\infty} A^k\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

From Theorem 5.7.1, we know that

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

must be a probability vector, so $c_1 = 1/3$. Thus, for any probability vector $\vec{x}$, we have proved the surprising fact that $A^k\vec{x}$ converges to $1/3\vec{v}_1$ as $k \to \infty$.

Or maybe you already know the following definitions and theorem and are not-at-all surprised. That is also possible.

Definition 5.7.3 *For a transition matrix, A, a* ***steady-state vector*** *is a probability vector, $\vec{x}$, such that $A\vec{x} = \vec{x}$.*

Example 5.7.2 Again, recall the preceding pizza problem had transition matrix

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

The probability vector

$$\vec{x} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

is a steady-state vector because $A\vec{x} = \vec{x}$.

Definition 5.7.4 *A transition matrix, A, is called* ***regular*** *if A^k has no zero entries for some positive integer k.*

A transition matrix being regular is equivalent to being able to get from any state in a transition diagram to any other state by following paths in the diagram. A Markov chain from a regular transition matrices has very predictable long-term behavior:

Theorem 5.7.3 *If $A \in \mathcal{M}_{n\times n}$ is a regular transition matrix, then there is a unique steady-state vector, $\vec{x}_0$, such that for any probability vector, $\vec{x} \in \mathbb{R}^n$, we have*

$$\lim_{k\to\infty} A^k\vec{x} = \vec{x}_0.$$

There's a lot going on in Theorem 5.7.3. For regular transition matrices, Markov chains starting at *any* probability vector all converge to the same unique steady-state vector. That is exactly what happened in Example 5.7.1; note that the transition matrix, A, is regular (because it already has no zero entries).

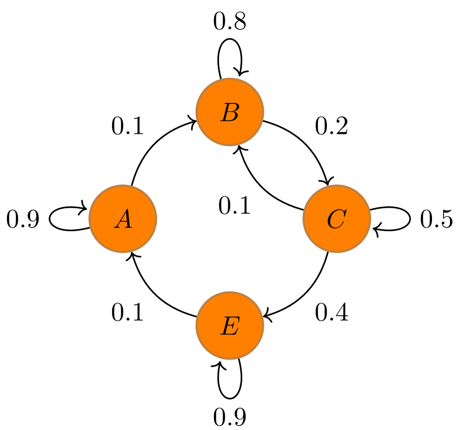
The proof of Theorem 5.7.3 is hard, and we don't want to do it. It's the end of the book. Give us a break!

Exploration 157 The Office of Bureaucracy and Mismanagement has three queues for filing paperwork. Properly filed paperwork requires the queues to be done consecutively; that is, when a hapless citizen delivers their paperwork at the front of Queue A, they must then wait in Queue B to deliver paperwork there; then they do the same in Queue C; then, and only then, can they escape. Here are some more fun facts about the OBM:

- 90% of citizens attempting to file paperwork in Queue A are asked to go to the back of the line and wait in Queue A again; the remaining 10% move on to Queue B.

- 80% filing in Queue B are told to go to the back of their line, and 20% move on to Queue C.
- The bureaucrats running Queue C are particularly cruel; while 50% of filers must go to the back of Queue C, 10% must go to the back of *Queue B*; the rest escape.
- 90% of citizens that escape the Office of Bureaucracy and Mismanagement never return; the remaining 10% go back to Queue A.

Here are the transition diagram and matrix:



$$A = \begin{bmatrix} 0.9 & 0 & 0 & 0.1 \\ 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.4 & 0.9 \end{bmatrix}$$

What percent of people eventually escape the Office of Bureaucracy and Mismanagement?

Exploration 158 The overlords at the Office of Bureaucracy and Mismanagement were dissatisfied with the insufficient level of misery they created, so some changes were implemented.[47] As a result, now 100% of people that escape never return. Nothing else changed. Show that the resulting transition matrix is not regular but that everyone eventually escapes.

47: That's right! Get those citizens in some queues!

As much fun as we've had with pizza and bureaucracy, it should definitely be noted that Markov processes have an incredible variety of uses, from flight scheduling to internet search engines. The possibilities are only limited by one's imagination. For more information, we refer you to said internet search engines.

Rank k Approximation

Here comes a handy technique. First, let's ruin your day with a big, awkward matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You can check that A has singular values $\sqrt{10}$, $\sqrt{6}$, and 1. Thinking of the singular value decomposition, $A = UDV^T$, where $U = [\vec{u}_1 \cdots \vec{u}_4]$ and $V = [\vec{v}_1 \cdots \vec{v}_7]$ are orthogonal and

$$D = \begin{bmatrix} \sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that since $A = UDV^T$, we could, inspired by the spectral decomposition for square matrices, write a singular value decomposition similarly.

Theorem 5.7.4 *If $A \in \mathcal{M}_{m\times n}$ has singular value decomposition $A = UDV^T$ with $U = [\vec{u}_1 \cdots \vec{u}_m]$, $V = [\vec{v}_1 \cdots \vec{v}_n]$, and singular values $\sigma_1, \ldots, \sigma_m$, then*

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_k \vec{u}_m \vec{v}_m^T.$$

Proof of this theorem comes from direct calculation, so let's skip that[48] and just look at it in the context of our matrix A.

48:  Aaarrgg!

$$A = \sqrt{10}\vec{u}_1\vec{v}_1^T + \sqrt{6}\vec{u}_2\vec{v}_2^T + \vec{u}_3\vec{v}_3^T + 0\vec{u}_4\vec{v}_4^T.$$

How much does that last term contribute to what the matrix A does to vectors? Literally nothing at all, but the third term doesn't contribute much either. Since $\sqrt{10}$ and $\sqrt{6}$ are much larger than the other singular values, 1 and 0, much more scaling is done by the first two terms of this decomposition. In this case, $\vec{v}_1$ and $\vec{v}_2$ are right-singular vectors for $\sqrt{10}$ and $\sqrt{6}$, respectively, so most of the linear transforming done by A is in the directions of $\vec{v}_1$ and $\vec{v}_2$. Working this way, we can feasibly restrict our attention to a linear transformation on a dimension two subspace that behaves a lot like A on a dimension seven space. In fact, all we have to do is cut off that last term and define

$$A_0 = \sqrt{10}\vec{u}_1\vec{v}_1^T + \sqrt{6}\vec{u}_2\vec{v}_2^T.$$

We can calculate $\vec{v}_1 = \vec{e}_1$ and $\vec{v}_2 = (-2\vec{e}_2 + \vec{e}_3\vec{e}_3)/\sqrt{6}$ and note that

$$\begin{aligned} A\vec{v}_1 &= \sqrt{10}\vec{u}_1 \text{ and} \\ A\vec{v}_2 &= \sqrt{6}\vec{u}_2 \end{aligned}$$

to see

$$A_0 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, one can *see* that A and A_0 are very much alike. Moreover, one can test $A\vec{x}$ and $A_0\vec{x}$ with a variety of vectors, $\vec{x}$, to see how similarly they behave as linear transformations.[49]

49:  Do it!

While A_0 is a reasonable approximation for A, A_0 is still quite large; it's the exact same size as A! Let's see what we can do about that. Note that

$$A_0 = \sqrt{10}\vec{u}_1\vec{v}_1^T + \sqrt{6}\vec{u}_2\vec{v}_2^T = [\vec{u}_1\ \vec{u}_2]\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}[\vec{v}_1\ \vec{v}_2]^T,$$

and we can rewrite this as

$$A_0[\vec{v}_1\ \vec{v}_2] = [\vec{u}_1\ \vec{u}_2]\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}.$$

Would you believe that

$$A[\vec{v}_1\ \vec{v}_2] = [\vec{u}_1\ \vec{u}_2]\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

as well? Let's check.

$$\begin{aligned} A[\vec{v}_1\ \vec{v}_2] &= \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2/\sqrt{6} \\ 0 & 1/\sqrt{6} \\ 0 & 1/\sqrt{6} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -\sqrt{6} \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = [\vec{u}_1\ \vec{u}_2]\begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{bmatrix}. \end{aligned}$$

When we define A_0 by cutting off that last term in the singular value decomposition of A, what's actually happening is that we're restricting A to the projection of the domain of A onto the span of $\vec{v}_1$ and $\vec{v}_2$. This sounds a lot like a theorem. First, we'll have a definition to formalize this "cutting off" procedure.

Definition 5.7.5 *If $A \in \mathcal{M}_{\times n}$ has singular value decomposition $A = UDV^T$ with $U = [\vec{u}_1 \cdots \vec{u}_m]$, $V = [\vec{v}_1 \cdots \vec{v}_n]$, and singular values $\sigma_1, \ldots, \sigma_m$, then for any positive integer $k \le m$, a* ***rank k approximation of*** *A is*

$$A_k = \sigma_1\vec{u}_1\vec{v}_1^T + \cdots + \sigma_k\vec{u}_k\vec{v}_k^T.$$

Theorem 5.7.5 *If $A \in \mathcal{M}_{m\times n}$ has rank k approximation, $A_k = \sigma_1\vec{u}_1\vec{v}_1^T + \cdots + \sigma_k\vec{u}_k\vec{v}_k^T$, then for any positive integer $k \le m$, A_k is the composition of A with the projection onto $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. In particular,*

$$A_k = [\vec{u}_1 \cdots \vec{u}_k]\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix}[\vec{v}_1 \cdots \vec{v}_k]^T = A[\vec{v}_1 \cdots \vec{v}_k][\vec{v}_1 \cdots \vec{v}_k]^T.$$

Moreover,

$$A[\vec{v}_1 \cdots \vec{v}_k] = [\sigma_1\vec{u}_1 \cdots \sigma_k\vec{u}_k].$$

PROOF. The first equality comes from matrix multiplication. The second comes from the singular value decomposition for A and Theorem 5.5.4, which states that $[\vec{v}_1 \cdots \vec{v}_k][\vec{v}_1 \cdots \vec{v}_k]^T$ is the projection onto $\mathrm{Span}\,\{\vec{v}_1, \ldots, \vec{v}_k\}$. The last inequality comes from the second after multiplication by $[\vec{v}_1 \cdots \vec{v}_k]$ and noting that $[\vec{v}_1 \cdots \vec{v}_k]^T[\vec{v}_1 \cdots \vec{v}_k] = I_k$. $\square$

Theorem 5.7.5 is particularly useful because

$$\begin{aligned} A &\in \mathcal{M}_{m\times n} \text{ and} \\ A[\vec{v}_1 \cdots \vec{v}_k] &\in \mathcal{M}_{m\times k}, \end{aligned}$$

and we could potentially choose k to be substantially smaller than n. [50]

50: Wait. Is that the end?

Looks like it is. I guess we should go home?

I like it here. I think I'll stay.

But what'll we do?

. . .