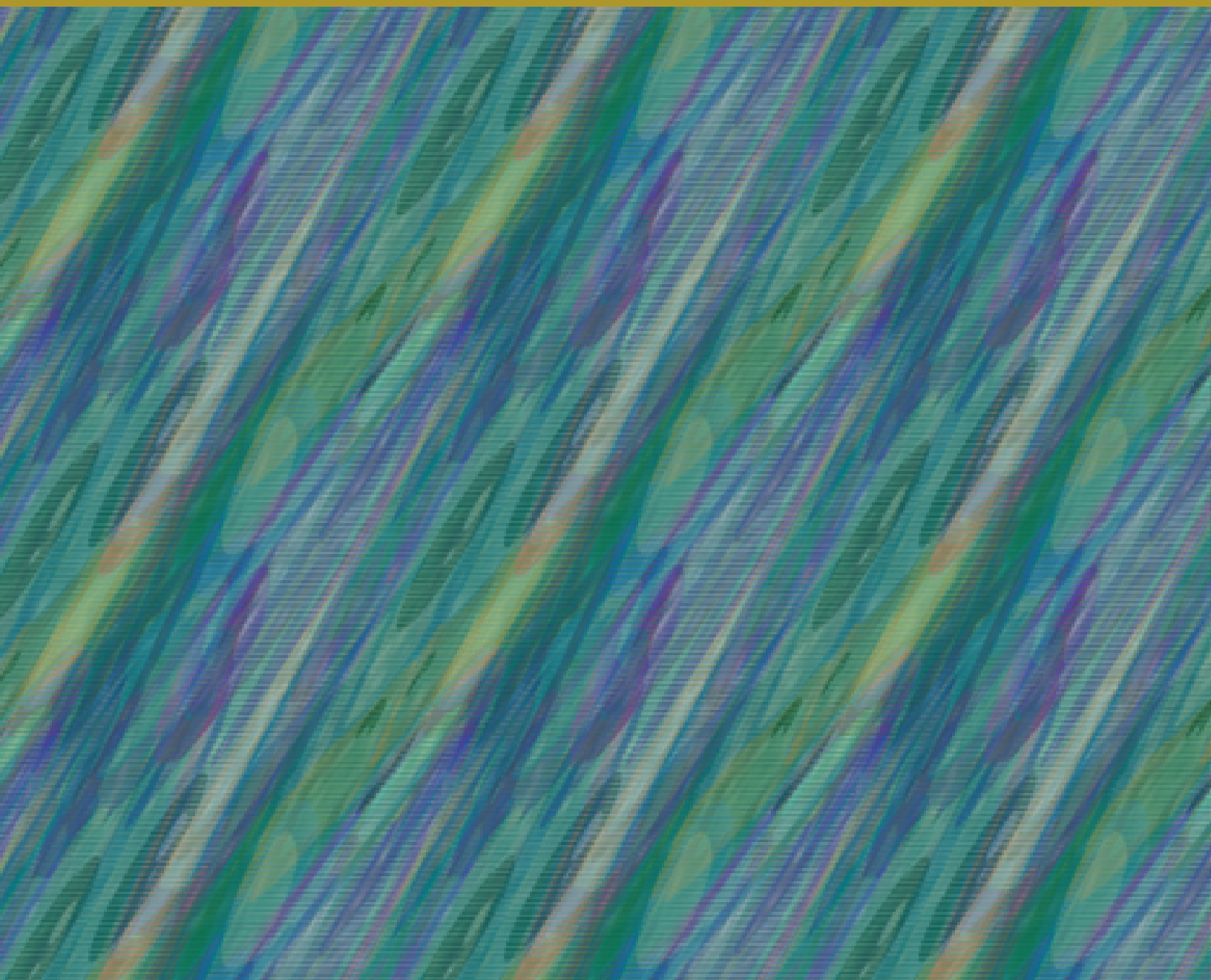


# LINEAR TRANSFORMATIONS ON VECTOR SPACES



Scott Kaschner and Amber Russell



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This is stupid. A pegasus can fly. An alicorn can fly.

Everyone knows that *unicorns can't fly*.

– anonymous linear algebra student

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# Preface

Welcome to Linear Algebra! Don't worry so much about unicorns right now. Let us first focus on the Linear Algebra part. Unicorns will, of course, appear later on naturally.

## To All Readers

This text was originally written for the Linear Algebra course offered at Butler University, but we have worked to make it appropriate for use elsewhere or for self-study. The order of the topics differs from many other Linear Algebra texts, but this was what we found worked best to help our students gain a deep understanding of the topic. We encourage you to follow the sections in order. We start with a quick, preliminary discussion of sets and functions, since these are central to the course. Then Chapters 1 and 2 focus on vector spaces. In Chapter 3, we study linear transformations, beginning with the definition and building to the connection with matrices. Finally, Chapters 4 and 5 are devoted to matrix theory.

If you would prefer to start the course with a discussion of matrices, we suggest the first part of Section 3.4, followed by Section 4.1 and all but the last subsection of Section 4.2. This covers the definition of matrices and their connections to solving systems of equations. One could then pick up at the beginning of the text and use the techniques just learned whenever solving a system of equations.

Some of the sections can be viewed as optional. In particular, Sections 1.5, 2.6, 3.6, 4.7, and 4.8 can be omitted without causing problems when covering future sections. After completing Sections 5.1, 5.2, and 5.3, each of Sections 5.4, 5.5, and 5.6 could serve independently as a capstone section with some additional applications in Section 5.7.

The text is written to be read. The tone is conversational (and sometimes a little silly); there is a unicorn theme, so don't be surprised to see pictures of them throughout. The mathematics is written thoroughly, with most results proven in either the section or the Appendix. There are many examples and "explorations" to engage readers as they encounter the material, written specifically with the idea that the reader work through the explorations as they read. We leave it to the instructor's discretion whether to provide solutions or work some of these during class meetings. We've also attempted to make it lighthearted and fun in places because there's no reason it shouldn't be. We hope you enjoy reading it, but mostly, we hope you learn from it.

## A Note about Unicorns

While you may have heard other myths about how to summon a unicorn, we're happy to reveal what happened to work for us. To summon a unicorn, you should include one in an example while writing a mathematics textbook. Apparently, unicorns love the study of mathematics so much, they reveal themselves to any textbook authors who appear to be receptive to their input. The ones who approached us initially were Ricky and Bubbles. We decided a nice way to include these muses was to allow them to narrate our side notes. They did get a bit carried away adding side notes of their own. Hopefully, you will not find them too distracting. It turns out it is difficult to *unsummon* a unicorn.

## A Message for Students

Here are some goals to keep in mind when using this book:

- **Learn the Content.** Linear Algebra is inarguably one of the most applicable areas in modern mathematics. It is foundational to advanced mathematics courses, but is also widely used in statistics, computer science, physics, chemistry, and a host of other fields. Note here that we have used the word “applicable” rather than “applied.” You are not learning an applied version of Linear Algebra. The goal of this text is to give you a strong foundation in this topic so that you can recognize the applications in your own field as you encounter them. Thus, we begin with the conceptual definitions, but we build towards the application side as the book progresses.
- **Improve Independent Learning Skills.** Realistically, this is the overall goal of undergraduate education. Yes, you will learn specific content in courses you take, but there will always be things you still need to learn as you continue on beyond coursework. The main goal of college is to prepare you for life after college, and a large part of that is giving you the tools to tackle challenges and master new concepts outside of a classroom setting. This may be a goal in all your classes, but it is not necessarily always addressed directly. This text helps facilitate this goal by giving you ways to interact as you read. We encourage you to re-work examples and also attempt all the explorations as you go through the reading. Uncovering areas in which your conceptual understanding can be refined is a valuable step in the learning process.
- **Transition to Advanced Coursework.** Linear Algebra is a prerequisite for many upper level mathematics courses and also courses in other departments. While some of this is due to its content, part of this prerequisite is also the experience of the course itself. For many students, this content will push you to think about more abstract mathematics topics than you may have in your previous experiences, and it will help you to see connections between different areas of mathematics, particularly algebra and geometry.

This content will be both challenging and time consuming, but it will also be rewarding. We hope you each find joy in the learning of this mathematics.

Best wishes,

Drs. Kaschner and Russell

# Contents

Preface	ii
Chapter 0. Functions on Sets	1
0.1 Sets	1
0.2 Functions	7
Chapter 1. Vector Spaces	14
1.1 Vector Spaces	14
1.2 Arrow Vectors and $\mathbb{R}^n$ for Small $n$	29
1.3 Linear Independence and Span	40
1.4 Subspaces	56
1.5 A Menagerie of Vector Spaces	71
Chapter 2. Bases	76
2.1 Introduction to Bases	76
2.2 More Fun with Bases	90
2.3 Coordinates, Inner Products, and Orthogonality: Oh my!	100
2.4 Orthogonal Sets	114
2.5 The Gram-Schmidt Process	129
2.6 Least Squares Applications	139
Chapter 3. Linear Transformations	147
3.1 More Fun with Functions	147
3.2 Linear Transformations	162
3.3 One-to-one and Onto Linear Transformations	177
3.4 Matrices	190
3.5 The Matrix of a Linear Transformation	204
3.6 More Fun with Linear Transformations	217
3.7 Applications of Linear Transformations	228
Chapter 4. More Fun with Matrices	232
4.1 Systems of Equations and Matrices	232
4.2 More Systems of Equations and Matrices	247
4.3 Matrix Techniques	259
4.4 Matrix Operations	274
4.5 Invertible Matrices	289
4.6 Matrix Theorems	300
4.7 More Fun with Least Squares	312
4.8 Another Graphics Application	320
Chapter 5. Square Matrices and Invariant Subspaces	324
5.1 Eigenvalues and How to Find Them	324
5.2 Determinants and More Fun with Eigenvalues	337
5.3 Diagonalization	350
5.4 Jordan Canonical Form	361



5.5 Spectral Theory	379
5.6 Singular Value Decomposition	388
5.7 Applications of Invariant Subspaces	395
Appendix	403
Additional Proofs	403
Answers to Selected Parts of Selected Explorations	410
Answers to Selected Exercises	413
Glossary	416